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Connection between non-Abelian tensor gauge fields and open strings

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Abstract

We compare the structure of the tree-level scattering amplitudes in non-Abelian tensor gauge field theory and in open string theory with Chan–Paton charges. We limit ourselves to considering only lower-rank tensor fields in both theories. We identify the symmetric and antisymmetric components of the second-rank tensor gauge field with the string excitations on the second and third exited levels. In the process of this identification we select only those parts of the tree-level scattering amplitudes in the open string theory which are linear in momenta and are dominant at low energies.

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1. Introduction

An infinite tower of particles of high spin naturally appears in the spectrum of different string theories. In the low-energy limit, the massless states of the open string theory with Chan–Paton charges can be identified with the Yang–Mills gauge quanta [1–3]. It is also expected that in the tensionless limit or, what is equivalent, at high energy and fixed angle scattering the string spectrum becomes effectively massless [4–18], and it is of great importance to identify these states as states of some Lagrangian quantum field theory [19–25].

One can imagine that massless states of tensionless string are combined into an infinite tower of non-Abelian tensor fields, and one could guess that the corresponding Lagrangian quantum field theory should be described by the extension of the Yang–Mills theory to the non-Abelian tensor gauge field theory. This possibility has recently been suggested in [27–29]. Recall that non-Abelian tensor gauge fields are defined as rank-(s+1) tensor gauge fields $A^a_{\mu\lambda_1\ldots\lambda_s}$ [1] and that one can construct an infinite series of forms $\mathcal{L}_s (s = 1, 2, \ldots)$ which

[1] Tensor gauge fields $A^a_{\mu\lambda_1\ldots\lambda_s}(x)$, $s = 0, 1, 2, \ldots$, are totally symmetric with respect to the indices $\lambda_1 \ldots \lambda_s$. *A priori* the tensor fields have no symmetries with respect to the first index μ. In particular, we have $A^a_{\mu\lambda} \neq A^a_{\lambda\mu}$ and $A^a_{\mu\lambda\rho} = A^a_{\mu\rho\lambda} \neq A^a_{\lambda\mu\rho}$. The adjoint group index $a = 1, \ldots, N^2 - 1$ in the case of the $SU(N)$ gauge group.

are invariant with respect to the extended gauge transformations [27–29]. These forms $\mathcal{L}_s$ are quadratic in the field strength tensors $G^a_{\mu\nu,\lambda_1\ldots\lambda_s}$, and the Lagrangian $\mathcal{L}$ is an infinite sum of these forms (1.1).

The gauge-invariant Lagrangian $\mathcal{L}$ defines *cubic and quartic interactions* with a *dimensionless coupling constant*[2] between charged gauge quanta [27–29]

$$A^a_\mu, \qquad A^a_{\mu\lambda_1}, \qquad A^a_{\mu\lambda_1\lambda_2}, \ldots$$

carrying a spin larger than 1. Note that all non-Abelian tensor gauge bosons have the same isotopic charges as the vector gauge boson. The gauge-invariant Lagrangian describing dynamical tensor gauge bosons of all ranks has the form

$$\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + g_2\mathcal{L}_2 + g_3\mathcal{L}_3 + \cdots, \tag{1.1}$$

where $\mathcal{L}_{\mathrm{YM}}$ is the Yang–Mills Lagrangian. For the lower-rank tensor gauge fields the Lagrangian has the following form [27–29]:

$$\begin{aligned} \mathcal{L}_{\mathrm{YM}} &= -\tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}, \\ \mathcal{L}_2 &= -\tfrac{1}{4}G^a_{\mu\nu,\lambda}G^a_{\mu\nu,\lambda} - \tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu,\lambda\lambda} + \tfrac{1}{4}G^a_{\mu\nu,\lambda}G^a_{\mu\lambda,\nu} + \tfrac{1}{4}G^a_{\mu\nu,\nu}G^a_{\mu\lambda,\lambda} + \tfrac{1}{2}G^a_{\mu\nu}G^a_{\mu\lambda,\nu\lambda}, \end{aligned} \tag{1.2}$$

where the generalized field strength tensors are

$$\begin{aligned} G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \\ G^a_{\mu\nu,\lambda} &= \partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda} + g f^{abc}\left(A^b_\mu A^c_{\nu\lambda} + A^b_{\mu\lambda} A^c_\nu\right), \\ G^a_{\mu\nu,\lambda\rho} &= \partial_\mu A^a_{\nu\lambda\rho} - \partial_\nu A^a_{\mu\lambda\rho} + g f^{abc}\left(A^b_\mu A^c_{\nu\lambda\rho} + A^b_{\mu\lambda} A^c_{\nu\rho} + A^b_{\mu\rho} A^c_{\nu\lambda} + A^b_{\mu\lambda\rho} A^c_\nu\right). \end{aligned} \tag{1.3}$$

The Lagrangian forms $\mathcal{L}_s$ for higher-rank fields can be found in the previous publications [27–29]. The above expressions define interacting gauge field theory with infinite many non-Abelian tensor gauge fields. Not much is known about the physical properties of such gauge field theory, and this paper is one in a series of papers devoted to this problem [30–34].

In the present paper, we shall focus our attention on the lower-rank tensor gauge field $A^a_{\mu\lambda}$, which decomposes in this theory into the symmetric tensor T_S of helicity two and the antisymmetric tensor T_A of helicity zero charged gauge bosons [29]. The Feynman rules for these propagating modes and their interaction vertices can be extracted from the above Lagrangian (1.2) [29] and are reviewed in the following section. These Feynman rules allow us to calculate tree-level scattering amplitudes for processes involving non-Abelian tensor gauge bosons [33, 34].

An interesting question is regarding whether or not non-Abelian tensor gauge fields can be identified with the states of open string theory with Chan–Paton charges [35][3]. Indeed, in the spectrum of the open string theory with Chan–Paton charges, there is a massless vector gauge boson V on the first exited level, and there are rank-two massive tensor bosons T_S on the second level and T_A on the third level carrying the same isotopic charges as the vector boson V. These states are depicted schematically in figure 1 as T_S and T_A. The emission vertices for these states are defined as follows [2, 3, 36]:

$$\begin{aligned} &e_\alpha(k) : \dot{X}^\alpha\, \mathrm{e}^{\mathrm{i}kX} : && \alpha' k^2 = 0 \\ &\varepsilon_{\alpha\alpha'}(k) : \dot{X}^\alpha \dot{X}^{\alpha'}\, \mathrm{e}^{\mathrm{i}kX} : && \alpha' k^2 = -1 \\ &\zeta_{\alpha\alpha'}(k)\tfrac{1}{2} : (\ddot{X}^\alpha \dot{X}^{\alpha'} - \ddot{X}^{\alpha'} \dot{X}^\alpha)\, \mathrm{e}^{\mathrm{i}kX} : && \alpha' k^2 = -2, \end{aligned} \tag{1.4}$$

and allow us to calculate different tree-level scattering amplitudes involving tensor bosons T_S and T_A.

[2] In D dimensions the coupling constant has dimension $(4 - D)/2$.

[3] This question was raised to the author by Costas Bachas and initiated this investigation.

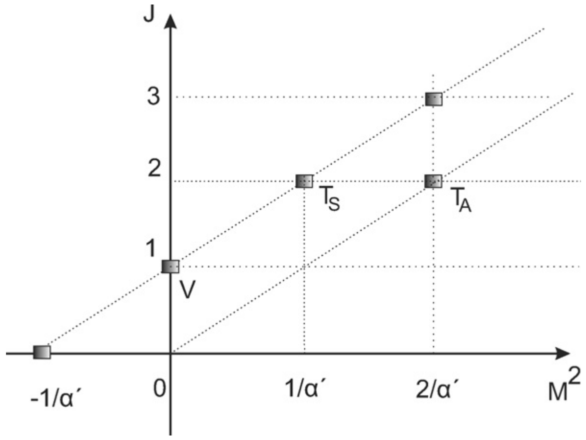

Figure 1. The excited levels of the open bosonic string. The state at the second level is a symmetric, traceless, rank-2 tensor T_S of $SO(D-1)$. At the third excited level, there is an antisymmetric rank-2 tensor T_A, of $SO(D-1)$. If the open string carries Chan–Paton charges at its endpoints then these excited states have the same isotopic charges as the massless vector boson V on the first excited level.

Our intention in this paper is to compare the tree-level scattering amplitudes of the second-rank tensor gauge bosons in non-Abelian tensor gauge field theory and tree-level scattering amplitudes of the tensor bosons in open string theory with Chan–Paton charges.

Our aim is twofold: first, to review the general structure of the interaction vertices and tree-level amplitudes in the non-Abelian tensor gauge field theory [27–29] and, then, to calculate *similar* tree-level scattering amplitudes in the open string theory in order to compare their structures. By 'similar' tree-level scattering amplitudes we mean the amplitudes of the open string theory which have coupling constants of the same dimensionality as the coupling constant of tensor gauge field theory in D dimensions. These amplitudes should have only one spacetime derivative in the case of triple interaction vertices and have no derivatives in the case of quartic interactions. This subsector of amplitudes appears in the low-energy limit when one can ignore the higher derivative terms, and is a natural candidate for comparison with the tree-level amplitudes in non-Abelian tensor gauge field theory [27–29]. It seems that this subclass of tree-level scattering amplitudes may provide important information about the structure of the open string theory.

The open string tree-level scattering amplitudes are defined on the mass-shell [2, 3], and in order to compare them with the tensor gauge field theory vertices we have to project vertices to the mass-shell. We have found that these scattering amplitudes in both theories have the same structure. This result tells that most probably the subsector of the open string theory is equivalent to the non-Abelian tensor gauge field theory.

2. Feynman rules for non-Abelian tensor gauge fields

We shall describe here the interaction vertices of the vector and tensor gauge bosons in non-Abelian tensor gauge field theory [27–29]. For that let us recapitulate the construction of the corresponding Feynman rules [29]. Because the Lagrangian (1.1), (1.2) is quadratic in field strength tensors $\mathcal{L} \sim (G)^2 = (\mathrm{d}A + g[A, A])^2$, it allows only *cubic and quartic interactions* with the *dimensionless coupling constant*:

$$g[A, A]\mathrm{d}A, \qquad g^2[A, A][A, A].$$

In particular, the interactions of the Yang–Mills vector bosons with the charged tensor gauge bosons described by the Lagrangian (1.2) are of this type. The Lagrangian (1.2) can be represented in the form of polynomial in the Yang–Mills vector field A^a_α and the tensor gauge field of the second rank $A^a_{\alpha\acute{\alpha}}$ [29]:

$$\mathcal{L}_2 = \frac{1}{2}A^a_{\alpha\acute{\alpha}}H_{\alpha\acute{\alpha}\gamma\acute{\gamma}}A^a_{\gamma\acute{\gamma}} + \frac{1}{2!}\mathcal{V}^{\mathrm{abc}}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}A^a_{\alpha\acute{\alpha}}A^b_\beta A^c_{\gamma\acute{\gamma}} + \frac{1}{2!2!}\mathcal{V}^{\mathrm{abcd}}_{\alpha\beta\gamma\acute{\gamma}\delta\acute{\delta}}A^a_\alpha A^b_\beta A^c_{\gamma\acute{\gamma}}A^d_{\delta\acute{\delta}} + \cdots. \tag{2.1}$$

It has the kinetic term AHA for the tensor gauge field, and the interaction vertices between two tensors and a vector, the VTT vertex $\mathcal{V}^{\mathrm{abc}}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}$ and two tensors and two vectors, the VVTT-vertex $\mathcal{V}^{\mathrm{abcd}}_{\alpha\beta\gamma\acute{\gamma}\delta\acute{\delta}}$. The kinetic operator of the Lagrangian is

$$\begin{aligned}H_{\alpha\acute{\alpha}\gamma\acute{\gamma}}(k) = &\left(-\eta_{\alpha\gamma}\eta_{\acute{\alpha}\acute{\gamma}} + \tfrac{1}{2}\eta_{\alpha\acute{\gamma}}\eta_{\acute{\alpha}\gamma} + \tfrac{1}{2}\eta_{\alpha\acute{\alpha}}\eta_{\gamma\acute{\gamma}}\right)k^2 + \eta_{\alpha\gamma}k_{\acute{\alpha}}k_{\acute{\gamma}} + \eta_{\acute{\alpha}\acute{\gamma}}k_\alpha k_\gamma\\ &- \tfrac{1}{2}(\eta_{\alpha\acute{\gamma}}k_{\acute{\alpha}}k_\gamma + \eta_{\acute{\alpha}\gamma}k_\alpha k_{\acute{\gamma}} + \eta_{\alpha\acute{\alpha}}k_\gamma k_{\acute{\gamma}} + \eta_{\gamma\acute{\gamma}}k_\alpha k_{\acute{\alpha}}),\end{aligned} \tag{2.2}$$

and it is symmetric under simultaneous interchange of the indices $\alpha \leftrightarrow \acute{\alpha}$ and $\gamma \leftrightarrow \acute{\gamma}$, but it is not symmetric with respect to a single interchange $\alpha \leftrightarrow \acute{\alpha}$ or $\gamma \leftrightarrow \acute{\gamma}$, because the tensor field $A^a_{\alpha\acute{\alpha}}$ is not a symmetric tensor. It is also a fully gauge-invariant operator $k_\alpha H_{\alpha\acute{\alpha}\gamma\acute{\gamma}} = 0, k_{\acute{\alpha}}H_{\alpha\acute{\alpha}\gamma\acute{\gamma}} = 0$; therefore, there are no negative norm states in the spectrum [27–29]. It describes the propagation of massless particles with helicities two and zero. Indeed, when k_μ is aligned along the third axis, $k_\mu = (k, 0, 0, k)$, the equation,

$$H_{\alpha\acute{\alpha}\gamma\acute{\gamma}}(k)f^{\gamma\acute{\gamma}}(k) = 0, \tag{2.3}$$

has three independent solutions of the helicities two and zero:

$$\varepsilon^1_{\alpha\acute{\alpha}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0,0, & 0,0\\ 0,1, & 0,0\\ 0,0, & -1,0\\ 0,0, & 0,0\end{pmatrix}, \qquad \varepsilon^2_{\alpha\acute{\alpha}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0,0,0,0\\ 0,0,1,0\\ 0,1,0,0\\ 0,0,0,0\end{pmatrix}, \qquad \zeta_{\alpha\acute{\alpha}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0, & 0,0,0\\ 0, & 0,1,0\\ 0, & -1,0,0\\ 0, & 0,0,0\end{pmatrix}, \tag{2.4}$$

with the property that $\varepsilon^1_{\gamma\acute{\gamma}}\varepsilon^1_{\lambda\acute{\lambda}} + \varepsilon^2_{\gamma\acute{\gamma}}\varepsilon^2_{\lambda\acute{\lambda}} \simeq \frac{1}{2}(\eta_{\gamma\lambda}\eta_{\acute{\gamma}\acute{\lambda}} + \eta_{\gamma\acute{\lambda}}\eta_{\acute{\gamma}\lambda} - \eta_{\gamma\acute{\gamma}}\eta_{\lambda\acute{\lambda}})$ and $\zeta_{\gamma\acute{\gamma}}\zeta_{\lambda\acute{\lambda}} \simeq \frac{1}{2}(\eta_{\gamma\lambda}\eta_{\acute{\gamma}\acute{\lambda}} - \eta_{\gamma\acute{\lambda}}\eta_{\acute{\gamma}\lambda})$. The symbol $\simeq$ means that the equation holds up to longitudinal terms. The second-rank tensor gauge field $A_{\alpha\acute{\alpha}}$ with 16 components describes in this theory three physical transversal polarizations. The propagator $\Delta_{\gamma\acute{\gamma}\lambda\acute{\lambda}}(k)$ is defined through the equation $H^{\mathrm{fix}}_{\alpha\acute{\alpha}\gamma\acute{\gamma}}(k)\Delta^{\gamma\acute{\gamma}}_{\lambda\acute{\lambda}}(k) = -\mathrm{i}\eta_{\alpha\lambda}\eta_{\acute{\alpha}\acute{\lambda}}$, and has the following form:

$$\Delta_{\gamma\acute{\gamma}\lambda\acute{\lambda}}(k) = -\mathrm{i}\frac{\pi_{\gamma\acute{\gamma}\lambda\acute{\lambda}}}{k^2 - \mathrm{i}\varepsilon}. \tag{2.5}$$

The corresponding residue can be represented as a sum:

$$\pi_{\gamma\acute{\gamma}\lambda\acute{\lambda}} = +(\eta_{\gamma\lambda}\eta_{\acute{\gamma}\acute{\lambda}} + \eta_{\gamma\acute{\lambda}}\eta_{\acute{\gamma}\lambda} - \eta_{\gamma\acute{\gamma}}\eta_{\lambda\acute{\lambda}}) + \tfrac{1}{3}(\eta_{\gamma\lambda}\eta_{\acute{\gamma}\acute{\lambda}} - \eta_{\gamma\acute{\lambda}}\eta_{\acute{\gamma}\lambda}). \tag{2.6}$$

The first term describes the $\lambda = \pm 2$ helicity states and is represented by the symmetric part $\varepsilon_{\alpha\acute{\alpha}}$ of the polarization tensor; the second term describes the $\lambda = 0$ helicity state and is represented by its antisymmetric part $\zeta_{\alpha\acute{\alpha}}$.

Let us now consider the three-particle interaction vertex VTT. There are two terms in $\mathcal{L}_2$ contributing to the vertex VTT. The first three-linear term of the Lagrangian (1.1) has the following form:

$$-\tfrac{1}{2}gf^{abc}\left(\partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda}\right)\left(A^b_\mu A^c_{\nu\lambda} + A^b_{\mu\lambda}A^c_\nu\right) - \tfrac{1}{4}gf^{abc}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)2A^b_{\mu\lambda}A^c_{\nu\lambda}.$$

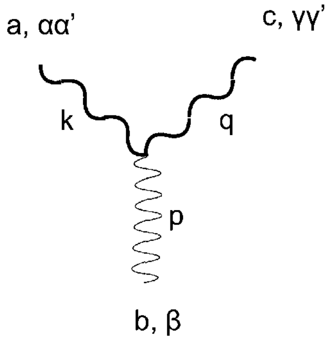

Figure 2. The interaction vertex for the vector gauge boson V and two tensor gauge bosons T—the VTT vertex—$\mathcal{V}^{\rm abc}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}(k,p,q)$ in non-Abelian tensor gauge field theory [29]. Vector gauge bosons are conventionally drawn as thin wave lines, tensor gauge bosons are thick wave lines. The Lorentz indices $\alpha\acute{\alpha}$ and momentum k belong to the first tensor gauge boson, the $\gamma\acute{\gamma}$ and momentum q belong to the second tensor gauge boson, and Lorentz index β and momentum p belong to the vector gauge boson.

This is in addition to the standard Yang–Mills VVV three-vector boson interaction vertex:

$$\mathcal{L}_1^{\rm cubic} = -\tfrac{1}{2} g f^{abc}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^b_\mu A^c_\nu,$$

which in the momentum representation has the form

$$\mathcal{V}^{abc}_{\alpha\beta\gamma}(k,p,q) = -\mathrm{i}gf^{abc}F_{\alpha\beta\gamma}(k,p,q) = -\mathrm{i}gf^{abc}[\eta_{\alpha\beta}(p-k)_\gamma + \eta_{\alpha\gamma}(k-q)_\beta + \eta_{\beta\gamma}(q-p)_\alpha]. \quad (2.7)$$

In momentum space, the first contribution has the form

$$-\mathrm{i}gf^{abc}F_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}(k,p,q) = -\mathrm{i}gf^{abc}[\eta_{\alpha\beta}(p-k)_\gamma + \eta_{\alpha\gamma}(k-q)_\beta + \eta_{\beta\gamma}(q-p)_\alpha]\eta_{\acute{\alpha}\acute{\gamma}}. \quad (2.8)$$

The second three-linear term of the Lagrangian (1.1) has the following form:

$$\begin{aligned} &+\tfrac{1}{2} g f^{abc}\left(\partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda}\right)\left(A^b_\mu A^c_{\lambda\nu} + A^b_{\mu\nu}A^c_\lambda\right) \\ &+\tfrac{1}{2} g f^{abc}\left(\partial_\mu A^a_{\nu\nu} - \partial_\nu A^a_{\mu\nu}\right)\left(A^b_\mu A^c_{\lambda\lambda} + A^b_{\mu\lambda}A^c_\lambda\right) \\ &+\tfrac{1}{2} g f^{abc}\left(\partial_\mu A^a_{\nu} - \partial_\nu A^a_{\mu}\right)\left(A^b_{\mu\nu} A^c_{\lambda\lambda} + A^b_{\mu\lambda}A^c_{\lambda\nu}\right), \end{aligned} \quad (2.9)$$

so that in the momentum space we have

$$\begin{aligned} \mathrm{i}g\tfrac{1}{2}f^{abc}F'_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}(k,p,q) = \mathrm{i}g\tfrac{1}{2}f^{abc}[&+(p-k)_\gamma(\eta_{\alpha\acute{\gamma}}\eta_{\acute{\alpha}\beta} + \eta_{\alpha\acute{\alpha}}\eta_{\beta\acute{\gamma}}) \\ &+(k-q)_\beta(\eta_{\alpha\acute{\gamma}}\eta_{\acute{\alpha}\gamma} + \eta_{\alpha\acute{\alpha}}\eta_{\gamma\acute{\gamma}}) + (q-p)_\alpha(\eta_{\acute{\alpha}\gamma}\eta_{\beta\acute{\gamma}} + \eta_{\acute{\alpha}\beta}\eta_{\gamma\acute{\gamma}}) \\ &+(p-k)_{\acute{\alpha}}\eta_{\alpha\beta}\eta_{\gamma\acute{\gamma}} + (p-k)_{\acute{\gamma}}\eta_{\alpha\beta}\eta_{\acute{\alpha}\gamma} + (k-q)_{\acute{\alpha}}\eta_{\alpha\gamma}\eta_{\beta\acute{\gamma}} + (k-q)_{\acute{\gamma}}\eta_{\alpha\gamma}\eta_{\acute{\alpha}\beta} \\ &+(q-p)_{\acute{\alpha}}\eta_{\beta\gamma}\eta_{\alpha\acute{\gamma}} + (q-p)_{\acute{\gamma}}\eta_{\alpha\acute{\alpha}}\eta_{\beta\gamma}]. \end{aligned} \quad (2.10)$$

Collecting two terms of the three-linear vertex VTT together we shall get [29]

$$\mathcal{V}^{\rm abc}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}(k,p,q) = -\mathrm{i}gf^{abc}\left\{F_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}} - \tfrac{1}{2}F'_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}\right\} \equiv -\mathrm{i}gf^{abc}\mathcal{F}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}, \quad (2.11)$$

where the indices $(a, \alpha, \acute{\alpha}, k)$ belong to the tensor gauge boson, (b, β, p) to the vector gauge boson and $(c, \gamma, \acute{\gamma}, q)$ to the second tensor gauge boson (see figure 2).

Let us consider now the four-particle interaction terms of the Lagrangian (1.2). We have the standard four-vector-boson-interaction vertex VVVV:

$$\begin{aligned} \mathcal{V}^{abcd}_{\alpha\beta\gamma\delta}(k,p,q,r) = &-g^2 f^{lac}f^{lbd}(\eta_{\alpha\beta}\eta_{\gamma\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}) - g^2 f^{lad}f^{lbc}(\eta_{\alpha\beta}\eta_{\gamma\delta} - \eta_{\alpha\gamma}\eta_{\beta\delta}) \\ &- g^2 f^{lab}f^{lcd}(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}) \end{aligned} \quad (2.12)$$

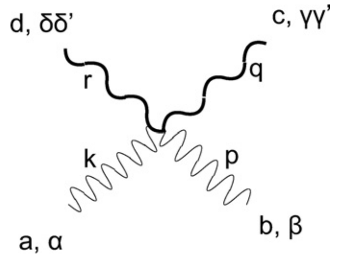

Figure 3. The quartic vertex with two vector gauge bosons and two tensor gauge bosons—the VVTT vertex—$\mathcal{V}^{abcd}_{\alpha\beta\gamma\acute{\gamma}\delta\acute{\delta}}(k,p,q,r)$ in non-Abelian tensor gauge field theory [29]. Vector gauge bosons are conventionally drawn as thin wave lines, and tensor gauge bosons are thick wave lines. The Lorentz indices $\gamma\acute{\gamma}$ and momentum q belong to the first tensor gauge boson, $\delta\acute{\delta}$ and momentum r belong to the second tensor gauge boson, the index α and momentum k belong to the first vector gauge boson and Lorentz index β and momentum p belong to the second vector gauge boson.

and a new interaction of two vector and two tensor gauge bosons—the VVTT vertex,

$$-\tfrac{1}{4}g^2 f^{abc} f^{a\acute{b}\acute{c}}\big(A^b_\mu A^c_{\nu\lambda} + A^b_{\mu\lambda}A^c_\nu\big)\big(A^{\acute{b}}_\mu A^{\acute{c}}_{\nu\lambda} + A^{\acute{b}}_{\mu\lambda}A^{\acute{c}}_\nu\big) - \tfrac{1}{2}g^2 f^{abc} f^{a\acute{b}\acute{c}} A^b_\mu A^c_\nu A^{\acute{b}}_{\mu\lambda} A^{\acute{c}}_{\nu\lambda}, \tag{2.13}$$

which in the momentum space will take the following form:

$$\begin{aligned} F^{abcd}_{\alpha\beta\gamma\acute{\gamma}\delta\acute{\delta}}(k,p,q,r) &= -g^2 f^{lac} f^{lbd}(\eta_{\alpha\beta}\eta_{\gamma\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma})\eta_{\acute{\gamma}\acute{\delta}} - g^2 f^{lad} f^{lbc}(\eta_{\alpha\beta}\eta_{\gamma\delta} - \eta_{\alpha\gamma}\eta_{\beta\delta})\eta_{\acute{\gamma}\acute{\delta}} \\ &\quad - g^2 f^{lab} f^{lcd}(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma})\eta_{\acute{\gamma}\acute{\delta}}. \end{aligned} \tag{2.14}$$

The second part of the vertex VVTT is

$$\begin{aligned} &+\tfrac{1}{4}g^2 f^{abc} f^{a\acute{b}\acute{c}}\big(A^b_\mu A^c_{\nu\lambda} + A^b_{\mu\lambda}A^c_\nu\big)\big(A^{\acute{b}}_\mu A^{\acute{c}}_{\lambda\nu} + A^{\acute{b}}_{\mu\nu}A^{\acute{c}}_\lambda\big) \\ &+\tfrac{1}{4}g^2 f^{abc} f^{a\acute{b}\acute{c}}\big(A^b_\mu A^c_{\nu\nu} + A^b_{\mu\nu}A^c_\nu\big)\big(A^{\acute{b}}_\mu A^{\acute{c}}_{\lambda\lambda} + A^{\acute{b}}_{\mu\lambda}A^{\acute{c}}_\lambda\big) \\ &+\tfrac{1}{2}g^2 f^{abc} f^{a\acute{b}\acute{c}} A^b_\mu A^c_\nu\big(A^{\acute{b}}_{\mu\nu}A^{\acute{c}}_{\lambda\lambda} + A^{\acute{b}}_{\mu\lambda}A^{\acute{c}}_{\lambda\nu}\big), \end{aligned} \tag{2.15}$$

which in the momentum representation will take the form

$$\begin{aligned} \acute{F}^{abcd}_{\alpha\beta\gamma\acute{\gamma}\delta\acute{\delta}}(k,p,q,r) &= \tfrac{1}{2}g^2 f^{lac} f^{lbd}[+\eta_{\alpha\beta}(\eta_{\gamma\acute{\delta}}\eta_{\acute{\gamma}\delta} + \eta_{\gamma\acute{\gamma}}\eta_{\delta\acute{\delta}}) - \eta_{\beta\gamma}(\eta_{\alpha\acute{\delta}}\eta_{\acute{\gamma}\delta} + \eta_{\alpha\acute{\gamma}}\eta_{\delta\acute{\delta}}) \\ &\quad - \eta_{\alpha\delta}(\eta_{\beta\acute{\gamma}}\eta_{\gamma\acute{\delta}} + \eta_{\beta\acute{\delta}}\eta_{\gamma\acute{\gamma}}) + \eta_{\gamma\delta}(\eta_{\alpha\acute{\delta}}\eta_{\beta\acute{\gamma}} + \eta_{\alpha\acute{\gamma}}\eta_{\beta\acute{\delta}})] \\ \tfrac{1}{2}g^2 f^{lad} f^{lbc}&[+\eta_{\alpha\beta}(\eta_{\gamma\acute{\delta}}\eta_{\acute{\gamma}\delta} + \eta_{\gamma\acute{\gamma}}\eta_{\delta\acute{\delta}}) - \eta_{\alpha\gamma}(\eta_{\beta\acute{\delta}}\eta_{\acute{\gamma}\delta} + \eta_{\beta\acute{\gamma}}\eta_{\delta\acute{\delta}}) \\ &\quad - \eta_{\beta\delta}(\eta_{\alpha\acute{\gamma}}\eta_{\gamma\acute{\delta}} + \eta_{\alpha\acute{\delta}}\eta_{\gamma\acute{\gamma}}) + \eta_{\gamma\delta}(\eta_{\alpha\acute{\gamma}}\eta_{\beta\acute{\delta}} + \eta_{\alpha\acute{\delta}}\eta_{\beta\acute{\gamma}})] \\ \tfrac{1}{2}g^2 f^{lab} f^{lcd}&[+\eta_{\alpha\gamma}(\eta_{\beta\acute{\gamma}}\eta_{\delta\acute{\delta}} + \eta_{\beta\acute{\delta}}\eta_{\delta\acute{\gamma}}) - \eta_{\beta\gamma}(\eta_{\alpha\acute{\gamma}}\eta_{\delta\acute{\delta}} + \eta_{\alpha\acute{\delta}}\eta_{\delta\acute{\gamma}}) \\ &\quad - \eta_{\alpha\delta}(\eta_{\beta\acute{\delta}}\eta_{\gamma\acute{\gamma}} + \eta_{\beta\acute{\gamma}}\eta_{\gamma\acute{\delta}}) + \eta_{\beta\delta}(\eta_{\alpha\acute{\delta}}\eta_{\gamma\acute{\gamma}} + \eta_{\alpha\acute{\gamma}}\eta_{\gamma\acute{\delta}})]. \end{aligned} \tag{2.16}$$

The total vertex is

$$\mathcal{V}^{abcd}_{\alpha\beta\gamma\acute{\gamma}\delta\acute{\delta}}(k,p,q,r) = F^{abcd}_{\alpha\beta\gamma\acute{\gamma}\delta\acute{\delta}}(k,p,q,r) + \acute{F}^{abcd}_{\alpha\beta\gamma\acute{\gamma}\delta\acute{\delta}}(k,p,q,r). \tag{2.17}$$

In summary, we have the off-mass-shell Yang–Mills vertex VVV (2.7), the new vertex VTT (2.11) together with the Yang–Mills vertex VVVV (2.12) and the new vertex VVTT (2.17) (see figures 2, 3).

2.1. Three-point amplitudes

In order to compare these vertices with the corresponding tree-level amplitudes of the open string theory one should project them to the mass-shell, because the string amplitudes can be computed only on the mass-shell[4]. The three-point scattering amplitudes for massless particles on mass-shell are equal to zero for the real momenta (k, p, q), but if one allows complex momenta or a different spacetime signature [37–47, 50–52], then these matrix elements will have nontrivial behavior and will allow us to compare tree-level scattering amplitudes in both theories.

Thus multiplying the above VTT vertex (2.11) by the vector wavefunction $e^{\beta}(p)$ and tensor wavefunctions $f^{\alpha\acute{\alpha}}(k)$ and $f^{\gamma\acute{\gamma}}(q)$ we shall get the amplitude $\mathcal{V}fef$:

$$\begin{aligned}\mathcal{V}^{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}_{abc}(k,p,q)|_{\text{mass-shell}} = -\mathrm{i}gf^{abc}\big[&+(k-q)^{\beta}\big(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'} - \tfrac{1}{2}\eta^{\alpha\gamma'}\eta^{\alpha'\gamma}\big)\\ &+(q-p)^{\alpha}\big(\eta^{\beta\gamma}\eta^{\alpha'\gamma'} - \tfrac{1}{2}\eta^{\beta\gamma'}\eta^{\alpha'\gamma}\big) - \tfrac{1}{2}(q-p)^{\alpha'}\big(\eta^{\beta\gamma}\eta^{\alpha\gamma'} - \tfrac{1}{2}\eta^{\beta\gamma'}\eta^{\alpha\gamma}\big)\\ &+(p-k)^{\gamma}\big(\eta^{\alpha\beta}\eta^{\alpha'\gamma'} - \tfrac{1}{2}\eta^{\alpha'\beta}\eta^{\alpha\gamma'}\big) - \tfrac{1}{2}(p-k)^{\gamma'}\big(\eta^{\alpha\beta}\eta^{\alpha'\gamma} - \tfrac{1}{2}\eta^{\alpha'\beta}\eta^{\alpha\gamma}\big)\big].\end{aligned} \tag{2.18}$$

Here we have used the transversality of the wavefunctions (2.4):

$$p_{\beta}\,e^{\beta}(p) = 0, \qquad k_{\alpha}f^{\alpha\acute{\alpha}}(k) = k_{\acute{\alpha}}f^{\alpha\acute{\alpha}}(k) = 0, \qquad q_{\gamma}f^{\gamma\acute{\gamma}}(q) = q_{\acute{\gamma}}f^{\gamma\acute{\gamma}}(k) = 0, \tag{2.19}$$

and that they are traceless, $\mathrm{tr}\,f(k) = \mathrm{tr}\,f(q) = 0$. In (2.18) and in all subsequent formulae for mass-shell amplitudes we shall not show the wavefunctions. The tensor wavefunction $f_{\alpha\acute{\alpha}}$ is a sum of symmetric $\varepsilon_{\alpha\acute{\alpha}}$ and antisymmetric $\zeta_{\alpha\acute{\alpha}}$ parts (2.4).

We shell separate the parts of this vertex which are symmetric, T_S, and antisymmetric, T_A, with respect to the indices of the tensor field of $f_{\alpha\acute{\alpha}} = \varepsilon_{\alpha\acute{\alpha}} + \zeta_{\alpha\acute{\alpha}}$. The symmetric $\mathcal{V}\varepsilon e\varepsilon$ part of the amplitude is

$$\begin{aligned}-\mathrm{i}\tfrac{1}{4}gf^{abc}\big[&+(k-q)^{\beta}(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'} + \eta^{\alpha\gamma'}\eta^{\alpha'\gamma}) + \tfrac{1}{4}(q-p)^{\alpha}(\eta^{\beta\gamma}\eta^{\alpha'\gamma'} + \eta^{\beta\gamma'}\eta^{\alpha'\gamma})\\ &+\tfrac{1}{4}(q-p)^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'} + \eta^{\beta\gamma'}\eta^{\alpha\gamma}) + \tfrac{1}{4}(p-k)^{\gamma}(\eta^{\alpha\beta}\eta^{\alpha'\gamma'} + \eta^{\alpha'\beta}\eta^{\alpha\gamma'})\\ &+\tfrac{1}{4}(p-k)^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma} + \eta^{\alpha'\beta}\eta^{\alpha\gamma})\big].\end{aligned} \tag{2.20}$$

The antisymmetric $\mathcal{V}\zeta e\zeta$ part of the amplitude is

$$\begin{aligned}-\mathrm{i}\tfrac{3}{4}gf^{abc}\big[&+(k-q)^{\beta}(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'} - \eta^{\alpha\gamma'}\eta^{\alpha'\gamma}) + \tfrac{3}{4}(q-p)^{\alpha}(\eta^{\beta\gamma}\eta^{\alpha'\gamma'} - \eta^{\beta\gamma'}\eta^{\alpha'\gamma})\\ &-\tfrac{3}{4}(q-p)^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'} - \eta^{\beta\gamma'}\eta^{\alpha\gamma}) + \tfrac{3}{4}(p-k)^{\gamma}(\eta^{\alpha\beta}\eta^{\alpha'\gamma'} - \eta^{\alpha'\beta}\eta^{\alpha\gamma'})\\ &-\tfrac{3}{4}(p-k)^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma} - \eta^{\alpha'\beta}\eta^{\alpha\gamma})\big].\end{aligned} \tag{2.21}$$

The mixed symmetry part $\mathcal{V}\varepsilon e\zeta$ of the amplitude is

$$\begin{aligned}-\mathrm{i}\frac{3}{16}gf^{abc}[&+(q-p)^{\alpha}(\eta^{\alpha'\gamma'}\eta^{\beta\gamma} - \eta^{\alpha'\gamma}\eta^{\beta\gamma'}) + (q-p)^{\alpha'}(\eta^{\alpha\gamma'}\eta^{\beta\gamma} - \eta^{\alpha\gamma}\eta^{\beta\gamma'})\\ &+(p-k)^{\gamma}(\eta^{\alpha\beta}\eta^{\alpha'\gamma'} + \eta^{\alpha\gamma'}\eta^{\alpha'\beta}) - (p-k)^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma} + \eta^{\alpha\gamma}\eta^{\alpha'\beta})].\end{aligned} \tag{2.22}$$

The last vertex is symmetric under the interchange $(\alpha \leftrightarrow \alpha')$ and antisymmetric under $(\gamma \leftrightarrow \gamma')$. There is also the mixed symmetry part of the vertex which is antisymmetric in $(\alpha \leftrightarrow \alpha')$ and symmetric under $(\gamma \leftrightarrow \gamma')$ the $\mathcal{V}\zeta e\varepsilon$ amplitude:

$$\begin{aligned}-\mathrm{i}\frac{3}{16}gf^{abc}[&+(q-p)^{\alpha}(\eta^{\alpha'\gamma'}\eta^{\beta\gamma} + \eta^{\alpha'\gamma}\eta^{\beta\gamma'}) - (q-p)^{\alpha'}(\eta^{\alpha\gamma'}\eta^{\beta\gamma} + \eta^{\alpha\gamma}\eta^{\beta\gamma'})\\ &+(p-k)^{\gamma}(\eta^{\alpha\beta}\eta^{\alpha'\gamma'} - \eta^{\alpha\gamma'}\eta^{\alpha'\beta}) + (p-k)^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma} - \eta^{\alpha\gamma}\eta^{\alpha'\beta})].\end{aligned} \tag{2.23}$$

One can check that the sum of the above four terms (2.20)–(2.23) gives the total vertex (2.18).

[4] We shall derive the corresponding string amplitudes in the following section.

2.2. Complex deformation of momenta

The nontrivial three-point amplitudes can be defined if one considers complex momenta or the spacetime signature $\eta^{\mu\nu} = (- + - +)$. Then the momenta can be chosen as follows [46]:

$$k_1^\mu = (\omega, z, \mathrm{i}z, k), \qquad k_2^\mu = (\omega, -z, -\mathrm{i}z, k), \qquad k_3^\mu = (2\omega, 0, 0, 2k),$$

to fulfil the momentum conservation,

$$k_1 + k_2 = k_3.$$

All massless bosons are on mass-shell: $k_1^2 = k_2^2 = k_3^2 = 0$, $(\omega^2 = k^2)$ and

$$k_1 \cdot k_2 = k_2 \cdot k_3 = k_3 \cdot k_1 = 0.$$

Let us first consider the matrix element VVV for the vector gauge bosons. The polarization vectors are

$$e_1^+ = \frac{1}{\sqrt{2}}\left(\frac{z}{\omega}, 1, -\mathrm{i}, -\frac{z}{k}\right), \qquad e_2^+ = \frac{1}{\sqrt{2}}\left(-\frac{z}{\omega}, 1, -\mathrm{i}, \frac{z}{k}\right), \qquad e_3^- = \frac{1}{\sqrt{2}}(0, 1, \mathrm{i}, 0)$$

and are orthogonal to the corresponding momenta:

$$k_1 \cdot e_1^+ = 0, k_2 \cdot e_2^+ = 0, k_3 \cdot e_3^- = 0.$$

We can compute now the matrix element using trilinear vertex VVV (2.7):

$$\begin{aligned} M(+,+,-) &= F^{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) e^+_{\mu_1}(k_1) e^+_{\mu_2}(k_2) e^-_{\mu_3}(k_3) \\ &= -2e_1^+ \cdot e_2^+ k_1 \cdot e_3^- - 2e_2^+ \cdot e_3^- k_2 \cdot e_1^+ + 2e_3^- \cdot e_1^+ k_3 \cdot e_2^+ = 8\sqrt{2}z. \end{aligned} \tag{2.24}$$

And indeed, it is nonzero and grows linearly with momentum deformation z. Using the spinor representation of the momenta and polarization vectors [44],

$$k_{a\dot a} = \lambda_a \tilde\lambda_{\dot a}, \qquad e^+_{a\dot a} = \frac{\mu_a \tilde\lambda_{\dot a}}{\langle \mu, \lambda\rangle}, \qquad e^-_{a\dot a} = \frac{\lambda_a \tilde\mu_{\dot a}}{[\lambda, \mu]},$$

where

$$\lambda_a = \left(\sqrt{k^+}, \frac{k_x + \mathrm{i}k_y}{\sqrt{k^+}}\right), \qquad \tilde\lambda_{\dot a} = \left(\sqrt{k^+}, \frac{k_x - \mathrm{i}k_y}{\sqrt{k^+}}\right), \qquad k^+ = k_t + k_z,$$

one can see that

$$\langle 1, 2\rangle = \langle 2, 3\rangle = \langle 3, 1\rangle = 0,$$

but

$$[1, 2] = -4z, \quad [2, 3] = 2\sqrt{2}z, \quad [3, 1] = 2\sqrt{2}z,$$

and the amplitude (2.7) in the spinor representation is

$$M(+,+,-) = -\sqrt{2}\frac{[1, 2]^4}{[1, 2][2, 3][3, 1]}.$$

We can calculate now the trilinear vertex VTT matrix element using expressions (2.11), (2.20):

$$\begin{aligned} M(+2, +1, -2) &= \mathcal{F}^{\alpha\acute\alpha\beta\gamma\acute\gamma}(k_1, k_2, k_3) \varepsilon^+_{\alpha\acute\alpha}(k_1) e^+_\beta(k_2) \varepsilon^-_{\gamma\acute\gamma}(k_3) \\ &= 2(k_1 + k_3) \cdot e_2^+ \varepsilon_1^+ \cdot \varepsilon_3^- + (-k_3 - k_2) \cdot \varepsilon_1^+ \cdot \varepsilon_3^- \cdot e_2^+ + (k_2 - k_1) \cdot \varepsilon_3^- \cdot \varepsilon_1^+ \cdot e_2^+ \\ &= 12\sqrt{2}z, \end{aligned} \tag{2.25}$$

where $\varepsilon^+_{\alpha\acute\alpha}(k_1) = e^+_\alpha(k_1) e^+_{\acute\alpha}(k_1)$, $\varepsilon^-_{\gamma\acute\gamma}(k_3) = e^-_\gamma(k_3) e^-_{\acute\gamma}(k_3)$. It is nonzero and also grows linearly with momentum deformation z. Using the spinor representation, we shall get

$$M(+2, +1, -2) = -\frac{3\sqrt{2}}{4}\frac{[1, 2]^6}{[1, 2][2, 3]^3[3, 1]}. \tag{2.26}$$

2.3. The dual Lagrangian

For completeness let us also recall the expression for the dual Lagrangian which is defined as follows [31, 32]:

$$\tilde{\mathcal{L}}_2 = -\tfrac{1}{4}\tilde{G}^a_{\mu\nu,\lambda}\tilde{G}^a_{\mu\nu,\lambda} - \tfrac{1}{4}G^a_{\mu\nu}\tilde{G}^a_{\mu\nu,\lambda\lambda} + \tfrac{1}{4}\tilde{G}^a_{\mu\nu,\lambda}\tilde{G}^a_{\mu\lambda,\nu} + \tfrac{1}{4}\tilde{G}^a_{\mu\nu,\nu}\tilde{G}^a_{\mu\lambda,\lambda} + \tfrac{1}{2}G^a_{\mu\nu}\tilde{G}^a_{\mu\lambda,\nu\lambda}, \tag{2.27}$$

where the dual field strength tensors are

$$\begin{aligned}\tilde{G}^a_{\mu\nu,\lambda} &= \partial_\mu A^a_{\lambda\nu} - \partial_\nu A^a_{\lambda\mu} + gf^{abc}\left(A^b_\mu A^c_{\lambda\nu} + A^b_{\lambda\mu}A^c_\nu\right),\\ \tilde{G}^a_{\mu\nu,\lambda\rho} &= \left\{\partial_\mu\left(\tfrac{2}{3}A^a_{\lambda\nu\rho} + \tfrac{2}{3}A^a_{\rho\nu\lambda} - \tfrac{1}{3}A^a_{\nu\lambda\rho}\right) + gf^{abc}A^b_\mu\left(\tfrac{2}{3}A^a_{\lambda\nu\rho} + \tfrac{2}{3}A^a_{\rho\nu\lambda} - \tfrac{1}{3}A^a_{\nu\lambda\rho}\right)\right.\\ &\quad \left. - \partial_\nu\left(\tfrac{2}{3}A^a_{\lambda\mu\rho} + \tfrac{2}{3}A^a_{\rho\mu\lambda} - \tfrac{1}{3}A^a_{\mu\lambda\rho}\right) + gf^{abc}\left(\tfrac{2}{3}A^a_{\lambda\mu\rho} + \tfrac{2}{3}A^a_{\rho\mu\lambda} - \tfrac{1}{3}A^a_{\mu\lambda\rho}\right)A^c_\nu\right\}\\ &\quad + gf^{abc}\left(A^b_{\lambda\mu}A^c_{\rho\nu} + A^b_{\rho\mu}A^c_{\lambda\nu}\right).\end{aligned} \tag{2.28}$$

Here we have a similar polynomial expansion of the dual Lagrangian:

$$\tilde{\mathcal{L}}_2 = \tfrac{1}{2}A^a_{\alpha\acute{\alpha}}\tilde{H}_{\alpha\acute{\alpha}\gamma\acute{\gamma}}A^a_{\gamma\acute{\gamma}} + \tfrac{1}{1!2!}\tilde{\mathcal{V}}^{abc}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}A^a_{\alpha\acute{\alpha}}A^b_\beta A^c_{\gamma\acute{\gamma}} + \tfrac{1}{2!2!}\tilde{\mathcal{V}}^{abcd}_{\alpha\beta\gamma\acute{\gamma}\delta\acute{\delta}}A^a_\alpha A^b_\beta A^c_{\gamma\acute{\gamma}}A^d_{\delta\acute{\delta}} + \cdots. \tag{2.29}$$

The kinetic term is identical to the kinetic term (2.2) of the Lagrangian $\mathcal{L}_2$:

$$\begin{aligned}\tilde{H}_{\alpha\acute{\alpha}\gamma\acute{\gamma}}(k) &= \left(-\eta_{\alpha\gamma}\eta_{\acute{\alpha}\acute{\gamma}} + \tfrac{1}{2}\eta_{\alpha\acute{\gamma}}\eta_{\acute{\alpha}\gamma} + \tfrac{1}{2}\eta_{\alpha\acute{\alpha}}\eta_{\gamma\acute{\gamma}}\right)k^2 + \eta_{\alpha\gamma}k_{\acute{\alpha}}k_{\acute{\gamma}} + \eta_{\acute{\alpha}\acute{\gamma}}k_\alpha k_\gamma\\ &\quad - \tfrac{1}{2}(\eta_{\alpha\acute{\gamma}}k_{\acute{\alpha}}k_\gamma + \eta_{\acute{\alpha}\gamma}k_\alpha k_{\acute{\gamma}} + \eta_{\alpha\acute{\alpha}}k_\gamma k_{\acute{\gamma}} + \eta_{\gamma\acute{\gamma}}k_\alpha k_{\acute{\alpha}})\end{aligned} \tag{2.30}$$

and is a fully gauge-invariant operator, $k_\alpha\tilde{H}_{\alpha\acute{\alpha}\gamma\acute{\gamma}} = 0, k_{\acute{\alpha}}\tilde{H}_{\alpha\acute{\alpha}\gamma\acute{\gamma}} = 0$. One can get also the explicit form of the dual cubic vertex VTT from the dual Lagrangian $\tilde{\mathcal{L}}_2$ (2.27):

$$\tilde{\mathcal{V}}^{abc}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}} = -\mathrm{i}gf^{abc}\left\{\tilde{F}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}} - \tfrac{1}{2}\tilde{F}'_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}\right\}, \tag{2.31}$$

where

$$\tilde{F}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}(k,p,q) = [\eta_{\acute{\alpha}\beta}(p-k)_{\acute{\gamma}} + \eta_{\acute{\alpha}\acute{\gamma}}(k-q)_\beta + \eta_{\beta\acute{\gamma}}(q-p)_{\acute{\alpha}}]\eta_{\alpha\gamma} \tag{2.32}$$

and

$$\begin{aligned}\tilde{F}'_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}(k,p,q) &= (p-k)_{\acute{\gamma}}(\eta_{\acute{\alpha}\gamma}\eta_{\alpha\beta} + \eta_{\alpha\acute{\alpha}}\eta_{\beta\gamma}) + (k-q)_\beta(\eta_{\alpha\acute{\gamma}}\eta_{\acute{\alpha}\gamma} + \eta_{\alpha\acute{\alpha}}\eta_{\gamma\acute{\gamma}})\\ &\quad + (q-p)_{\acute{\alpha}}(\eta_{\alpha\acute{\gamma}}\eta_{\beta\gamma} + \eta_{\alpha\beta}\eta_{\gamma\acute{\gamma}}) + (p-k)_\alpha\eta_{\acute{\alpha}\beta}\eta_{\gamma\acute{\gamma}} + (p-k)_\gamma\eta_{\acute{\alpha}\beta}\eta_{\alpha\acute{\gamma}}\\ &\quad + (k-q)_\alpha\eta_{\acute{\alpha}\acute{\gamma}}\eta_{\beta\gamma} + (k-q)_\gamma\eta_{\acute{\alpha}\acute{\gamma}}\eta_{\alpha\beta} + (q-p)_\alpha\eta_{\beta\acute{\gamma}}\eta_{\acute{\alpha}\gamma} + (q-p)_\gamma\eta_{\alpha\acute{\alpha}}\eta_{\beta\acute{\gamma}}.\end{aligned} \tag{2.33}$$

There is an important property of the dual vertex (2.31) which follows from the fact that the above Lagrangians (1.2) and (2.27) are dual to each other in the sense of the transformation:

$$\begin{aligned}\tilde{A}_{\alpha\acute{\alpha}} &= A_{\acute{\alpha}\alpha},\\ \tilde{A}_{\alpha\acute{\alpha}\acute{\acute{\alpha}}} &= \tfrac{2}{3}(A_{\acute{\alpha}\alpha\acute{\acute{\alpha}}} + A_{\acute{\acute{\alpha}}\alpha\acute{\alpha}}) - \tfrac{1}{3}A_{\alpha\acute{\alpha}\acute{\acute{\alpha}}}.\end{aligned} \tag{2.34}$$

Indeed, if one simultaneously interchanges the indices $\alpha \leftrightarrow \acute{\alpha}, \gamma \leftrightarrow \acute{\gamma}$ of the dual cubic vertex (2.31), then one can see that it will transform into the cubic vertex (2.11) and via versa:

$$\tilde{\mathcal{V}}^{abc}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}} = -\mathrm{i}gf^{abc}\left\{\tilde{F}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}} - \tfrac{1}{2}\tilde{F}'_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}\right\} = -\mathrm{i}gf^{abc}\left\{F_{\acute{\alpha}\alpha\beta\acute{\gamma}\gamma} - \tfrac{1}{2}F'_{\acute{\alpha}\alpha\beta\acute{\gamma}\gamma}\right\} = \mathcal{V}^{\mathrm{abc}}_{\acute{\alpha}\alpha\beta\acute{\gamma}\gamma}. \tag{2.35}$$

It is also obvious from the above relation that if one considers the *self-dual sum*,

$$\mathcal{L}_2 + \tilde{\mathcal{L}}_2, \tag{2.36}$$

then the corresponding VTT vertex will be self-dual in the sense that under the duality transformation (2.34) $\alpha \leftrightarrow \acute{\alpha}, \gamma \leftrightarrow \acute{\gamma}$ it will be mapped into itself:

$$\mathcal{V}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}} + \tilde{\mathcal{V}}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}} \to \mathcal{V}_{\acute{\alpha}\alpha\beta\acute{\gamma}\gamma} + \tilde{\mathcal{V}}_{\acute{\alpha}\alpha\beta\acute{\gamma}\gamma} = \tilde{\mathcal{V}}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}} + \mathcal{V}_{\alpha\acute{\alpha}\beta\gamma\acute{\gamma}}. \tag{2.37}$$

3. Open strings tree-level amplitudes

In this section, we shall calculate tree-level scattering amplitudes in the open string theory with Chan–Paton charges in order to compare them with the corresponding matrix elements in the non-Abelian tensor gauge field theory. We shall consider scattering amplitudes of the lower excited states of the open string depicted in figure 1, that is of the charged scalar, vector and tensor bosons. To set up notation, let us begin with the simplest example of the tree-level scattering amplitudes for the three on-shell massless vector bosons. The vertex operator has the following form [2, 3]:

$$e_\alpha(k) : \dot{X}^\alpha \, \mathrm{e}^{\mathrm{i}kX}(y): \tag{3.1}$$

and we shall represent the disc as the upper half-plane so that the boundary coordinate y is real: $y \in [-\infty, +\infty]$. The tree amplitude can be represented in the form

$$\mathcal{V}^{\mu_1\mu_2\mu_3}_{a_1a_2a_3}(k_1,k_2,k_3) = F^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}) + F^{\mu_2\mu_1\mu_3}(k_2,k_1,k_3)\,\mathrm{tr}(\lambda^{a_2}\lambda^{a_1}\lambda^{a_3}) :$$

where the matrix element F is given below:

$$F^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = \int \prod_i \mathrm{d}\mu(y_i) <: \dot{X}^{\mu_1}\,\mathrm{e}^{\mathrm{i}k_1X}(y_1) :: \dot{X}^{\mu_2}\,\mathrm{e}^{\mathrm{i}k_2X}(y_2) :: \dot{X}^{\mu_3}\,\mathrm{e}^{\mathrm{i}k_3X}(y_3) :>$$

$$= \lim_{y_1=0,y_2=1,y_3\to\infty} \prod_{i<j} |y_i - y_j|^{2\alpha' k_i k_j} y_3^2$$

$$\left\{ F^{\mu_1}_{y_1} F^{\mu_2}_{y_2} F^{\mu_2}_{y_2} - 2\alpha' \left[F^{\mu_1}_{y_1} \frac{\eta^{\mu_2\mu_3}}{(y_2-y_3)^2} + F^{\mu_2}_{y_2} \frac{\eta^{\mu_3\mu_1}}{(y_3-y_1)^2} + F^{\mu_3}_{y_3} \frac{\eta^{\mu_1\mu_2}}{(y_1-y_2)^2} \right] \right\},$$

and we have to sum over two orderings of the vertex operators on the disc. The vector functions F^μ_y are given below (3.4). The λ^a are isotopic matrices. All bosons are on mass-shell:

$$\alpha' k_1^2 = \alpha' k_2^2 = \alpha' k_3^2 = 0$$

and $k_1 + k_2 + k_3 = 0$. The wavefunctions of the vector bosons are

$$e_{\mu_1}(k_1), e_{\mu_2}(k_2), e_{\mu_3}(k_3) \tag{3.2}$$

and are transversal to the corresponding momenta $k_i \cdot e(k_i) = 0, i = 1, 2, 3$. The matrix element $F^{\mu_1\mu_2\mu_3}$ has the following dimensional structure:

$$[F^{\mu_1\mu_2\mu_3}] \sim (\alpha')^3(k)^3 + (\alpha')^2(k)^1. \tag{3.3}$$

It is not difficult to calculate both these terms, but our intention is to investigate only that part of the amplitude which is dominant at low energies, that is the last, linear in momentum term. We shall fix the integration measure by the choice $y_1 = 0$, $y_2 = 1$, $y_3 \to \infty$:

$$\lim_{y_1=0,y_2=1,y_3\to\infty} \prod_{i<j} |y_i - y_j|^{2\alpha' k_i k_j} \to 1$$

$$\begin{aligned}
F^{\mu_1}_{y_1} &= -2\mathrm{i}\alpha' \left(\frac{k_2^{\mu_1}}{y_1 - y_2} + \frac{k_3^{\mu_1}}{y_1 - y_3} \right) \to -2\mathrm{i}\alpha' \left(-k_2^{\mu_1}\right) \\
F^{\mu_2}_{y_2} &= -2\mathrm{i}\alpha' \left(\frac{k_1^{\mu_2}}{y_2 - y_1} + \frac{k_3^{\mu_2}}{y_2 - y_3} \right) \to -2\mathrm{i}\alpha' \left(+k_1^{\mu_2}\right) \\
F^{\mu_3}_{y_3} &= -2\mathrm{i}\alpha' \left(\frac{k_1^{\mu_3}}{y_3 - y_1} + \frac{k_2^{\mu_3}}{y_3 - y_2} \right) \to -2\mathrm{i}\alpha' \left(-\frac{k_1^{\mu_3}}{y_3^2} \right).
\end{aligned} \tag{3.4}$$

Thus for $F^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)\,\mathrm{tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})$ we have

$$\mathrm{i}(2\alpha')^2 \left[-k_2^{\mu_1}\eta^{\mu_2\mu_3} + k_1^{\mu_2}\eta^{\mu_3\mu_1} - k_1^{\mu_3}\eta^{\mu_1\mu_2} + 2\alpha' k_2^{\mu_1} k_1^{\mu_2} k_1^{\mu_3} \right] \mathrm{tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}) \tag{3.5}$$

and adding the equal term[5],

$$i(2\alpha')^2\big[+k_3^{\mu_1}\eta^{\mu_2\mu_3} - k_3^{\mu_2}\eta^{\mu_3\mu_1} + k_2^{\mu_3}\eta^{\mu_1\mu_2} + 2\alpha' k_2^{\mu_1}k_1^{\mu_2}k_1^{\mu_3}\big]\mathrm{tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}), \tag{3.6}$$

we can get the total matrix element together with the reversed cyclic orientation, $a_1, \mu_1, k_1 \leftrightarrow a_2, \mu_2, k_2$:

$$2i(\alpha')^2\big[(k_3-k_2)^{\mu_1}\eta^{\mu_2\mu_3} + (k_1-k_3)^{\mu_2}\eta^{\mu_3\mu_1} + (k_2-k_1)^{\mu_3}\eta^{\mu_1\mu_2} + 2\alpha'(k_2-k_3)^{\mu_1}k_1^{\mu_2}k_1^{\mu_3}\big]\mathrm{tr}([\lambda^{a_1},\lambda^{a_2}]\lambda^{a_3}). \tag{3.7}$$

Leaving only the linear in momentum term we shall have the corresponding VVV amplitude:

$$\mathcal{V}^{\mu_1\mu_2\mu_3}_{a_1a_2a_3}(k_1,k_2,k_3) = ig\,\mathrm{tr}([\lambda^{a_1},\lambda^{a_2}]\lambda^{a_3})[(k_3-k_2)^{\mu_1}\eta^{\mu_2\mu_3} + (k_1-k_3)^{\mu_2}\eta^{\mu_3\mu_1} + (k_2-k_1)^{\mu_3}\eta^{\mu_1\mu_2}], \tag{3.8}$$

which coincides with the Yang–Mills vertex (2.7) projected to the mass-shell.

In the following subsection, we shall perform a similar calculation of the scattering amplitude between two symmetric tensor bosons T_S and a vector boson V in open string theory (see figure 1) in order to compare it with the amplitude (2.18) in non-Abelian tensor gauge field theory.

3.1. Tree-level amplitudes for two symmetric tensors and vector

The vertex operator for the symmetric T_S rank-2 tensor boson on the third level is

$$\varepsilon_{\alpha\alpha'}(k) : \dot{X}^{\alpha}\dot{X}^{\alpha'}\,e^{ikX}(y) : \tag{3.9}$$

and together with the vertex (3.1) can be used to calculate now the scattering amplitude between vector and two tensor bosons:

$$\mathcal{V}^{\alpha\alpha'\beta\gamma\gamma'}_{abc}(k,p,q) = F^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q)\,\mathrm{tr}(\lambda^a\lambda^b\lambda^c) + F^{\gamma\gamma'\beta\alpha\alpha'}(q,p,k)\,\mathrm{tr}(\lambda^c\lambda^b\lambda^a), \tag{3.10}$$

where the wavefunctions of the pair of tensor gauge bosons and the vector boson are

$$\varepsilon_{\alpha\alpha'}(k), \qquad e_{\beta}(p), \qquad \varepsilon_{\gamma\gamma'}(q). \tag{3.11}$$

We shall define, for convenience, $k_1 \equiv k$, $k_2 \equiv q$, $k_3 \equiv p$, and $k_1+k_2+k_3 = 0$. The mass-shell conditions are

$$\alpha' k_1^2 = \alpha' k_2^2 = -1, \qquad \alpha' k_3^2 = 0.$$

We have to calculate the correlation function,

$$F_S^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q) = \int\prod_i d\mu(y_i) <: \dot{X}^{\alpha}\dot{X}^{\alpha'}\,e^{ikX}(y_1) :: \dot{X}^{\gamma}\dot{X}^{\gamma'}\,e^{iqX}(y_2) :: \dot{X}^{\beta}\,e^{ipX}(y_3) :>$$

$$= \lim_{y_1=0,y_2=1,y_3\to\infty}\prod_{i<j}|y_i-y_j|^{2\alpha' k_i k_j}y_3^2$$

$$\Big\{(F^{\alpha}F^{\alpha'})_{y_1}F^{\beta}_{y_3}(F^{\gamma}F^{\gamma'})_{y_2} - 2\alpha'\Big[F^{\alpha}_{y_1}F^{\beta}_{y_3}F^{\gamma}_{y_2}\frac{\eta^{\alpha'\gamma'}}{(y_1-y_2)^2} + F^{\alpha}_{y_1}F^{\beta}_{y_3}F^{\gamma'}_{y_2}\frac{\eta^{\alpha'\gamma}}{(y_1-y_2)^2}$$

$$+ F^{\alpha'}_{y_1}F^{\beta}_{y_3}F^{\gamma}_{y_2}\frac{\eta^{\alpha\gamma'}}{(y_1-y_2)^2} + F^{\alpha'}_{y_1}F^{\beta}_{y_3}F^{\gamma'}_{y_2}\frac{\eta^{\alpha\gamma}}{(y_1-y_2)^2} + F^{\alpha}_{y_1}F^{\alpha'}_{y_1}F^{\gamma}_{y_2}\frac{\eta^{\beta\gamma'}}{(y_3-y_2)^2}$$

$$+ F^{\alpha}_{y_1}F^{\alpha'}_{y_1}F^{\gamma'}_{y_2}\frac{\eta^{\beta\gamma}}{(y_3-y_2)^2} + F^{\alpha}_{y_1}F^{\gamma}_{y_2}F^{\gamma'}_{y_2}\frac{\eta^{\beta\alpha'}}{(y_3-y_1)^2} + F^{\alpha'}_{y_1}F^{\gamma}_{y_2}F^{\gamma'}_{y_2}\frac{\eta^{\beta\alpha}}{(y_3-y_1)^2}\Big]$$

$$+ (2\alpha')^2\Big[+F^{\beta}_{y_3}\frac{\eta^{\alpha\gamma}}{(y_1-y_2)^2}\frac{\eta^{\alpha'\gamma'}}{(y_1-y_2)^2} + F^{\beta}_{y_3}\frac{\eta^{\alpha\gamma'}}{(y_1-y_2)^2}\frac{\eta^{\alpha'\gamma}}{(y_1-y_2)^2}$$

[5] One should use momentum conservation and the transversality of the wavefunctions.

$$
\begin{aligned}
&+F^{\alpha}_{y_1}\frac{\eta^{\beta\gamma}}{(y_3-y_2)^2}\frac{\eta^{\alpha'\gamma'}}{(y_1-y_2)^2}+F^{\alpha}_{y_1}\frac{\eta^{\beta\gamma'}}{(y_3-y_2)^2}\frac{\eta^{\alpha'\gamma}}{(y_1-y_2)^2}\\
&+F^{\alpha'}_{y_1}\frac{\eta^{\beta\gamma}}{(y_3-y_2)^2}\frac{\eta^{\alpha\gamma'}}{(y_1-y_2)^2}+F^{\alpha'}_{y_1}\frac{\eta^{\beta\gamma'}}{(y_3-y_2)^2}\frac{\eta^{\alpha\gamma}}{(y_1-y_2)^2}\\
&+F^{\gamma}_{y_2}\frac{\eta^{\beta\alpha}}{(y_3-y_1)^2}\frac{\eta^{\alpha'\gamma'}}{(y_1-y_2)^2}+F^{\gamma}_{y_2}\frac{\eta^{\beta\alpha'}}{(y_3-y_1)^2}\frac{\eta^{\alpha\gamma'}}{(y_1-y_2)^2}\\
&+F^{\gamma'}_{y_2}\frac{\eta^{\beta\alpha}}{(y_3-y_1)^2}\frac{\eta^{\alpha'\gamma}}{(y_1-y_2)^2}+F^{\gamma'}_{y_2}\frac{\eta^{\beta\alpha'}}{(y_3-y_1)^2}\frac{\eta^{\alpha\gamma}}{(y_1-y_2)^2}\Bigg]\Bigg\}. \qquad (3.12)
\end{aligned}
$$

The amplitude has the following dimensional structure:

$$
F^{\alpha\alpha'\beta\gamma\gamma'}\sim(\alpha')^5(k)^5+(\alpha')^4(k)^3+(\alpha')^3(k)^1, \qquad (3.13)
$$

where the first term contains the fifth power of α' and the fifth power of momentum, and so on. We are interested in calculating only the last dominant term which is linear in momenta. We shall fix the integration measure by the choice $y_1 = 0$, $y_2 = 1$, $y_3 \to \infty$:

$$
\begin{aligned}
&\lim_{y_1=0,y_2=1,y_3\to\infty}\prod_{i<j}|y_i-y_j|^{2\alpha' k_i k_j}\to 1,\\
&F^{\alpha}_{y_1}=-2\mathrm{i}\alpha'\left(\frac{p^{\alpha}}{y_1-y_3}+\frac{q^{\alpha}}{y_1-y_2}\right)\to-2\mathrm{i}\alpha'(-q^{\alpha})\\
&F^{\gamma}_{y_2}=-2\mathrm{i}\alpha'\left(\frac{p^{\gamma}}{y_2-y_3}+\frac{k^{\gamma}}{y_2-y_1}\right)\to-2\mathrm{i}\alpha'(k^{\gamma}) \qquad (3.14)\\
&F^{\beta}_{y_3}=-2\mathrm{i}\alpha'\left(\frac{k^{\beta}}{y_3-y_1}+\frac{q^{\beta}}{y_3-y_2}\right)\to-2\mathrm{i}\alpha'\left(-\frac{k^{\beta}}{y_3^2}\right),
\end{aligned}
$$

and keeping only part of the amplitude linear in momentum we shall get

$$
\begin{aligned}
F_S^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q)=&-\mathrm{i}(2\alpha')^3[-k^{\beta}(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'}+\eta^{\alpha\gamma'}\eta^{\alpha'\gamma})-q^{\alpha}(\eta^{\beta\gamma}\eta^{\alpha'\gamma'}+\eta^{\beta\gamma'}\eta^{\alpha'\gamma})\\
&-q^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'}+\eta^{\beta\gamma'}\eta^{\alpha\gamma})+k^{\gamma}(\eta^{\alpha\beta}\eta^{\alpha'\gamma'}+\eta^{\alpha'\beta}\eta^{\alpha\gamma'})\\
&+k^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma}+\eta^{\alpha'\beta}\eta^{\alpha\gamma})]. \qquad (3.15)
\end{aligned}
$$

We can add an equal term,

$$
\begin{aligned}
F_S^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q)=&-\mathrm{i}(2\alpha')^3[+q^{\beta}(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'}+\eta^{\alpha\gamma'}\eta^{\alpha'\gamma})+p^{\alpha}(\eta^{\beta\gamma}\eta^{\alpha'\gamma'}+\eta^{\beta\gamma'}\eta^{\alpha'\gamma})\\
&+p^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'}+\eta^{\beta\gamma'}\eta^{\alpha\gamma})-p^{\gamma}(\eta^{\alpha\beta}\eta^{\alpha'\gamma'}+\eta^{\alpha'\beta}\eta^{\alpha\gamma'})\\
&-p^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma}+\eta^{\alpha'\beta}\eta^{\alpha\gamma})], \qquad (3.16)
\end{aligned}
$$

in order to get the symmetric expression:

$$
\begin{aligned}
F_S^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q)=&-\frac{\mathrm{i}}{2}(2\alpha')^3[+(q-k)^{\beta}(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'}+\eta^{\alpha\gamma'}\eta^{\alpha'\gamma})\\
&+(p-q)^{\alpha}(\eta^{\beta\gamma}\eta^{\alpha'\gamma'}+\eta^{\beta\gamma'}\eta^{\alpha'\gamma})+(p-q)^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'}+\eta^{\beta\gamma'}\eta^{\alpha\gamma})\\
&+(k-p)^{\gamma}(\eta^{\alpha\beta}\eta^{\alpha'\gamma'}+\eta^{\alpha'\beta}\eta^{\alpha\gamma'})+(k-p)^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma}+\eta^{\alpha'\beta}\eta^{\alpha\gamma})]. \qquad (3.17)
\end{aligned}
$$

Substituting this into expression (3.10) with the terms in the reversed cyclic orientation $a, (\alpha, \alpha'), k \leftrightarrow c, (\gamma, \gamma'), q$ we shall get

$$
\mathcal{V}_{abc}^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q)=\mathrm{tr}([\lambda^a,\lambda^b]\lambda^c)F_S^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q). \qquad (3.18)
$$

This expression should be compared with expression (2.20) in tensor gauge field theory. We see that they have identical Lorentz structure, but with some differences in the coefficients.

We do not know exactly the origin of this difference, but most probably it is connected with contributions of higher-rank non-Abelian tensor gauge fields, which we do not take into consideration in this paper.

3.2. Tree-level amplitudes for two antisymmetric tensors and vector

The vertex operator for the antisymmetric T_A rank-2 tensor boson on the third excited level is

$$\zeta_{\alpha\alpha'}(k) : \ddot{X}^{\{\alpha}\dot{X}^{\alpha'\}}\,\mathrm{e}^{\mathrm{i}kX} = \zeta_{\alpha\alpha'}(k)\tfrac{1}{2} : (\ddot{X}^{\alpha}\dot{X}^{\alpha'} - \ddot{X}^{\alpha'}\dot{X}^{\alpha})\,\mathrm{e}^{\mathrm{i}kX} : . \tag{3.19}$$

The antisymmetric wavefunctions of the pair of tensor gauge bosons and the vector boson are

$$\zeta_{\alpha\alpha'}(k), \quad \zeta_{\gamma\gamma'}(q), \quad e_{\beta}(p) \tag{3.20}$$

and we shall define for convenience $k_1 \equiv k, k_2 \equiv q, k_3 \equiv p,$. The mass-shell conditions are

$$\alpha' k_1^2 = \alpha' k_2^2 = -2, \alpha' k_3^2 = 0$$

and $k_1 + k_2 + k_3 = 0$. We have to calculate the correlation function:

$$F_A^{\alpha\alpha'\beta\gamma\gamma'}(k, p, q) \tag{3.21}$$

$$= \int \prod_i \mathrm{d}\mu(y_i) <: \ddot{X}^{\{\alpha}\dot{X}^{\alpha'\}}\,\mathrm{e}^{\mathrm{i}kX}(y_1) :: \ddot{X}^{\{\gamma}\dot{X}^{\gamma'\}}\,\mathrm{e}^{\mathrm{i}qX}(y_2) :: \dot{X}^{\beta}\,\mathrm{e}^{\mathrm{i}pX}(y_3) :> \tag{3.22}$$

$$= \lim_{y_1=0, y_2=1, y_3\to\infty} \prod_{i<j} |y_i - y_j|^{2\alpha' k_i k_j} y_3^2 \frac{1}{4}\{\mathcal{O}(\alpha')^5 (k)^5 + \mathcal{O}(\alpha')^4 (k)^3$$

$$+ (2\alpha')^2 F^{\beta}_{y_3}\left[+\frac{-6\eta^{\alpha\gamma}}{(y_1-y_2)^4}\frac{\eta^{\alpha'\gamma'}}{(y_1-y_2)^2} + \frac{-2\eta^{\alpha\gamma'}}{(y_1-y_2)^3}\frac{2\eta^{\alpha'\gamma}}{(y_1-y_2)^3}\right.$$

$$- \frac{-2\eta^{\alpha\gamma}}{(y_1-y_2)^3}\frac{2\eta^{\alpha'\gamma'}}{(y_1-y_2)^3} - \frac{-6\eta^{\alpha\gamma'}}{(y_1-y_2)^4}\frac{\eta^{\alpha'\gamma}}{(y_1-y_2)^2}$$

$$- \frac{2\eta^{\alpha\gamma}}{(y_1-y_2)^3}\frac{-2\eta^{\alpha'\gamma'}}{(y_1-y_2)^3} - \frac{\eta^{\alpha\gamma'}}{(y_1-y_2)^2}\frac{-6\eta^{\alpha'\gamma}}{(y_1-y_2)^4}$$

$$\left. + \frac{\eta^{\alpha\gamma}}{(y_1-y_2)^2}\frac{-6\eta^{\alpha'\gamma'}}{(y_1-y_2)^4} + \frac{2\eta^{\alpha\gamma'}}{(y_1-y_2)^3}\frac{-2\eta^{\alpha'\gamma}}{(y_1-y_2)^3}\right]$$

$$+ F^{\alpha}_{y_1}\left[-\frac{2\eta^{\beta\gamma}}{(y_3-y_2)^3}\frac{-2\eta^{\alpha'\gamma'}}{(y_1-y_2)^3} - \frac{\eta^{\beta\gamma'}}{(y_3-y_2)^2}\frac{-6\eta^{\alpha'\gamma}}{(y_1-y_2)^4}\right.$$

$$\left. + \frac{\eta^{\beta\gamma}}{(y_3-y_2)^2}\frac{-6\eta^{\alpha'\gamma'}}{(y_1-y_2)^4} + \frac{2\eta^{\beta\gamma'}}{(y_3-y_2)^3}\frac{-2\eta^{\alpha'\gamma}}{(y_1-y_2)^3}\right]$$

$$+ \dot{F}^{\alpha}_{y_1}\left[+\frac{2\eta^{\beta\gamma}}{(y_3-y_2)^3}\frac{\eta^{\alpha'\gamma'}}{(y_1-y_2)^2} + \frac{\eta^{\beta\gamma'}}{(y_3-y_2)^2}\frac{2\eta^{\alpha'\gamma}}{(y_1-y_2)^3}\right.$$

$$\left. - \frac{\eta^{\beta\gamma}}{(y_3-y_2)^2}\frac{2\eta^{\alpha'\gamma'}}{(y_1-y_2)^3} + \frac{2\eta^{\beta\gamma'}}{(y_3-y_2)^3}\frac{\eta^{\alpha'\gamma}}{(y_1-y_2)^2}\right]$$

$$+ F^{\alpha'}_{y_1}\left[+\frac{2\eta^{\beta\gamma}}{(y_3-y_2)^3}\frac{-2\eta^{\alpha\gamma'}}{(y_1-y_2)^3} + \frac{\eta^{\beta\gamma'}}{(y_3-y_2)^2}\frac{-6\eta^{\alpha\gamma}}{(y_1-y_2)^4}\right.$$

$$\left. - \frac{\eta^{\beta\gamma}}{(y_3-y_2)^2}\frac{-6\eta^{\alpha\gamma'}}{(y_1-y_2)^4} - \frac{2\eta^{\beta\gamma'}}{(y_3-y_2)^3}\frac{-2\eta^{\alpha\gamma}}{(y_1-y_2)^3}\right]$$

$$+ \dot{F}^{\alpha'}_{y_1}\left[-\frac{2\eta^{\beta\gamma}}{(y_3-y_2)^3}\frac{\eta^{\alpha\gamma'}}{(y_1-y_2)^2} - \frac{\eta^{\beta\gamma'}}{(y_3-y_2)^2}\frac{2\eta^{\alpha\gamma}}{(y_1-y_2)^3}\right.$$

$$+ \frac{\eta^{\beta\gamma}}{(y_3-y_2)^2}\frac{2\eta^{\alpha\gamma'}}{(y_1-y_2)^3} + \frac{2\eta^{\beta\gamma'}}{(y_3-y_2)^3}\frac{\eta^{\alpha\gamma}}{(y_1-y_2)^2}\Bigg]$$
$$+ F^{\gamma}_{y_2}[..] + \dot{F}^{\gamma}_{y_2}[..] + F^{\gamma'}_{y_2}[..] + \dot{F}^{\gamma'}_{y_2}[..]\}, \qquad (3.23)$$

where dots denote the terms which one can recover by interchanging α, α' with γ, γ'. This amplitude has the same dimensional structure as the symmetric one:

$$F_A^{\alpha\alpha'\beta\gamma\gamma'} \sim (\alpha')^5(k)^5 + (\alpha')^4(k)^3 + (\alpha')^3(k)^1, \qquad (3.24)$$

and we shall calculate the term which contains only the first power of momentum, that is, the last term. Taking the corresponding limit,

$$\lim_{y_1=0,y_2=1,y_3\to\infty} \prod_{i<j} |y_i - y_j|^{2\alpha' k_i k_j} \to 1$$
$$F^{\alpha}_{y_1} = -2i\alpha'\left(\frac{p^\alpha}{y_1-y_3} + \frac{q^\alpha}{y_1-y_2}\right) \to -2i\alpha'(-q^\alpha)$$
$$F^{\gamma}_{y_2} = -2i\alpha'\left(\frac{p^\gamma}{y_2-y_3} + \frac{k^\gamma}{y_2-y_1}\right) \to -2i\alpha'(k^\gamma)$$
$$F^{\beta}_{y_3} = -2i\alpha'\left(\frac{k^\beta}{y_3-y_1} + \frac{q^\beta}{y_3-y_2}\right) \to -2i\alpha'\left(-\frac{k^\beta}{y_3^2}\right) \qquad (3.25)$$
$$\dot{F}^{\alpha}_{y_1} = 2i\alpha'\left(\frac{p^\alpha}{(y_1-y_3)^2} + \frac{q^\alpha}{(y_1-y_2)^2}\right) \to -2i\alpha'(-q^\alpha)$$
$$\dot{F}^{\gamma}_{y_2} = 2i\alpha'\left(\frac{p^\gamma}{(y_2-y_3)^2} + \frac{k^\gamma}{(y_2-y_1)^2}\right) \to -2i\alpha'(-k^\gamma),$$

we shall get

$$F_A^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q) = -i(2\alpha')^3[+k^\beta(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'} - \eta^{\alpha\gamma'}\eta^{\alpha'\gamma}) + q^\alpha(\eta^{\beta\gamma}\eta^{\alpha'\gamma'} - \eta^{\beta\gamma'}\eta^{\alpha'\gamma})$$
$$- q^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'} - \eta^{\beta\gamma'}\eta^{\alpha\gamma}) + k^\gamma(\eta^{\alpha\beta}\eta^{\alpha'\gamma'} - \eta^{\alpha'\beta}\eta^{\alpha\gamma'})$$
$$- k^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma} - \eta^{\alpha'\beta}\eta^{\alpha\gamma})].$$

Adding the equal term,

$$F_A^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q) = -i(2\alpha')^3[-q^\beta(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'} - \eta^{\alpha\gamma'}\eta^{\alpha'\gamma}) - p^\alpha(\eta^{\beta\gamma}\eta^{\alpha'\gamma'} - \eta^{\beta\gamma'}\eta^{\alpha'\gamma})$$
$$+ p^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'} - \eta^{\beta\gamma'}\eta^{\alpha\gamma}) - p^\gamma(\eta^{\alpha\beta}\eta^{\alpha'\gamma'} - \eta^{\alpha'\beta}\eta^{\alpha\gamma'})$$
$$+ p^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma} - \eta^{\alpha'\beta}\eta^{\alpha\gamma})],$$

we shall get a symmetric in the momenta expression:

$$F_A^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q) = -\frac{i}{2}(2\alpha')^3[-(q-k)^\beta(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'} - \eta^{\alpha\gamma'}\eta^{\alpha'\gamma})$$
$$-(p-q)^\alpha(\eta^{\beta\gamma}\eta^{\alpha'\gamma'} - \eta^{\beta\gamma'}\eta^{\alpha'\gamma}) + (p-q)^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'} - \eta^{\beta\gamma'}\eta^{\alpha\gamma})$$
$$-(p-k)^\gamma(\eta^{\alpha\beta}\eta^{\alpha'\gamma'} - \eta^{\alpha'\beta}\eta^{\alpha\gamma'}) + (p-k)^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma} - \eta^{\alpha'\beta}\eta^{\alpha\gamma})]. \qquad (3.26)$$

Substituting this into expression (3.10) with the terms in the reversed cyclic orientation, $a, (\alpha, \alpha'), k \leftrightarrow c, (\gamma, \gamma'), q$, we shall finally get

$$\mathcal{V}_{abc}^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q) = \mathrm{tr}([\lambda^a, \lambda^b]\lambda^c) F_A^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q). \qquad (3.27)$$

This expression should be compared with expression (2.21) and again we see that they have identical structure.

3.3. Mixed symmetry amplitudes

Finally we shall calculate the amplitude between symmetric and antisymmetric tensor bosons, and a vector. The vertex operator for the symmetric T_S and antisymmetric T_A rank-2 tensor bosons has been given above, (3.9) and (3.19). Therefore, the matrix element is

$$
\begin{aligned}
&F_{SA}^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q)\\
&\quad= \int \prod_i \mathrm{d}\mu(y_i) <: \dot{X}^{\alpha}\dot{X}^{\alpha'}\,\mathrm{e}^{\mathrm{i}kX}(y_1) :: \ddot{X}^{\{\gamma}\dot{X}^{\gamma'\}}\,\mathrm{e}^{\mathrm{i}qX}(y_2) :: \dot{X}^{\beta}\,\mathrm{e}^{\mathrm{i}pX}(y_3) :>\\
&\quad= \lim_{y_1=0,y_2=1,y_3\to\infty} \prod_{i<j}|y_i-y_j|^{2\alpha' k_i k_j} y_3^2\frac{1}{2}\Bigg\{\mathcal{O}(\alpha')^5(k)^5+\mathcal{O}(\alpha')^4(k)^3\\
&\qquad+(2\alpha')^2F_{y_3}^{\beta}\left[+\frac{2\eta^{\alpha\gamma}}{(y_1-y_2)^3}\frac{\eta^{\alpha'\gamma'}}{(y_1-y_2)^2}+\frac{\eta^{\alpha\gamma'}}{(y_1-y_2)^2}\frac{2\eta^{\alpha'\gamma}}{(y_1-y_2)^3}\right.\\
&\qquad\left.-\frac{\eta^{\alpha\gamma}}{(y_1-y_2)^2}\frac{2\eta^{\alpha'\gamma'}}{(y_1-y_2)^3}-\frac{2\eta^{\alpha\gamma'}}{(y_1-y_2)^3}\frac{\eta^{\alpha'\gamma}}{(y_1-y_2)^2}\right]\\
&\qquad+F_{y_1}^{\alpha}\left[+\frac{-2\eta^{\beta\gamma}}{(y_2-y_3)^3}\frac{\eta^{\alpha'\gamma'}}{(y_1-y_2)^2}+\frac{\eta^{\beta\gamma'}}{(y_2-y_3)^2}\frac{2\eta^{\alpha'\gamma}}{(y_1-y_2)^3}\right.\\
&\qquad\left.-\frac{\eta^{\beta\gamma}}{(y_2-y_3)^2}\frac{2\eta^{\alpha'\gamma'}}{(y_1-y_2)^3}-\frac{-2\eta^{\beta\gamma'}}{(y_2-y_3)^3}\frac{\eta^{\alpha'\gamma}}{(y_1-y_2)^2}\right]\\
&\qquad+F_{y_1}^{\alpha'}\left[+\frac{-2\eta^{\beta\gamma}}{(y_2-y_3)^3}\frac{\eta^{\alpha\gamma'}}{(y_1-y_2)^2}+\frac{\eta^{\beta\gamma'}}{(y_2-y_3)^2}\frac{2\eta^{\alpha\gamma}}{(y_1-y_2)^3}\right.\\
&\qquad\left.-\frac{\eta^{\beta\gamma}}{(y_2-y_3)^2}\frac{2\eta^{\alpha\gamma'}}{(y_1-y_2)^3}-\frac{-2\eta^{\beta\gamma'}}{(y_2-y_3)^3}\frac{\eta^{\alpha\gamma}}{(y_1-y_2)^2}\right]\\
&\qquad+\dot{F}_{y_2}^{\gamma}\left[+\frac{\eta^{\alpha\gamma'}}{(y_1-y_2)^2}\frac{\eta^{\alpha'\beta}}{(y_1-y_3)^2}+\frac{\eta^{\alpha\beta}}{(y_1-y_3)^2}\frac{\eta^{\alpha'\gamma'}}{(y_1-y_2)^2}\right]\\
&\qquad+F_{y_2}^{\gamma}\left[-\frac{2\eta^{\alpha\gamma'}}{(y_1-y_2)^3}\frac{\eta^{\alpha'\beta}}{(y_1-y_3)^2}-\frac{\eta^{\alpha\beta}}{(y_1-y_3)^2}\frac{2\eta^{\alpha'\gamma'}}{(y_1-y_2)^3}\right]\\
&\qquad+F_{y_2}^{\gamma'}\left[+\frac{2\eta^{\alpha\gamma}}{(y_1-y_2)^3}\frac{\eta^{\alpha'\beta}}{(y_1-y_3)^2}+\frac{\eta^{\alpha\beta}}{(y_1-y_3)^2}\frac{2\eta^{\alpha'\gamma}}{(y_1-y_2)^3}\right]\\
&\qquad+\dot{F}_{y_2}^{\gamma'}\left[-\frac{\eta^{\alpha\gamma}}{(y_1-y_2)^2}\frac{\eta^{\alpha'\beta}}{(y_1-y_3)^2}-\frac{\eta^{\alpha\beta}}{(y_1-y_3)^2}\frac{\eta^{\alpha'\gamma}}{(y_1-y_2)^2}\right]\Bigg\}.
\end{aligned}
\tag{3.28}
$$

Taking the corresponding limit we shall get

$$
\begin{aligned}
F_{SA}^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q) &= -\mathrm{i}(2\alpha')^3[-q^{\alpha}(\eta^{\beta\gamma}\eta^{\alpha'\gamma'}-\eta^{\beta\gamma'}\eta^{\alpha'\gamma})-q^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'}-\eta^{\alpha\gamma}\eta^{\beta\gamma'})\\
&\quad+\tfrac{1}{2}k^{\gamma}(\eta^{\alpha\beta}\eta^{\alpha'\gamma'}+\eta^{\alpha\gamma'}\eta^{\alpha'\beta})-\tfrac{1}{2}k^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma}+\eta^{\alpha\gamma}\eta^{\alpha'\beta})]
\end{aligned}
$$

or, in equivalent form, as

$$
\begin{aligned}
F_{SA}^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q) &= -\frac{\mathrm{i}}{2}(2\alpha')^3[+(p-q)^{\alpha}(\eta^{\beta\gamma}\eta^{\alpha'\gamma'}-\eta^{\beta\gamma'}\eta^{\alpha'\gamma})\\
&\quad+(p-q)^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'}-\eta^{\alpha\gamma}\eta^{\beta\gamma'})+\frac{1}{2}(k-p)^{\gamma}(\eta^{\alpha\beta}\eta^{\alpha'\gamma'}+\eta^{\alpha\gamma'}\eta^{\alpha'\beta})\\
&\quad-\frac{1}{2}(k-p)^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma}+\eta^{\alpha\gamma}\eta^{\alpha'\beta})].
\end{aligned}
$$

The amplitude will take the following form:

$$F_{AS}^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q) = -\frac{\mathrm{i}}{2}(2\alpha')^3\Big[+\frac{1}{2}(p-q)^{\alpha}(\eta^{\beta\gamma}\eta^{\alpha'\gamma'}+\eta^{\beta\gamma'}\eta^{\alpha'\gamma}) - \frac{1}{2}(p-q)^{\alpha'}(\eta^{\beta\gamma}\eta^{\alpha\gamma'}+\eta^{\alpha\gamma}\eta^{\beta\gamma'}) + (k-p)^{\gamma}(\eta^{\alpha\beta}\eta^{\alpha'\gamma'}-\eta^{\alpha\gamma'}\eta^{\alpha'\beta}) + (k-p)^{\gamma'}(\eta^{\alpha\beta}\eta^{\alpha'\gamma}-\eta^{\alpha\gamma}\eta^{\alpha'\beta})\Big]. \quad (3.29)$$

Plugging both expressions into (3.9) with the terms in the reversed cyclic orientation $a, (\alpha, \alpha'), k \leftrightarrow c, (\gamma, \gamma'), q$ we shall get

$$\mathcal{V}_{abc}^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q) = \mathrm{tr}([\lambda^a,\lambda^b]\lambda^c)\big[F_{SA}^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q) + F_{AS}^{\alpha\alpha'\beta\gamma\gamma'}(k,p,q)\big], \quad (3.30)$$

which is in agreement with the expression we obtained in the tensor gauge field theory (2.22) and (2.23).

4. Scattering amplitude of two scalars and two tensors

We shall get less trivial information about the structure of the VTT vertex of the open string theory if we compute the four-particle scattering amplitude of two tachyon and two tensor bosons (see figure 4). In the low-energy limit, these amplitudes will be dominated by the exchange of the massless vector boson (see figure 5) and shall provide the information about the structure of cubic VTT vertex $\mathcal{V}_{1-2-2}$ and a new vertex between scalar, vector and a tensor $\mathcal{V}_{0-1-2}$. We are interested therefore in calculating the following scattering amplitude on the disc (see figure 4):

$$\begin{aligned} F^{\mu\nu,\lambda\rho}(k_1,k_2;k_3,k_4) &= \int\prod_i \mathrm{d}\mu(y_i) <: \mathrm{e}^{\mathrm{i}k_1X}(y_1) :: \mathrm{e}^{\mathrm{i}k_2X}(y_2) :: \dot{X}^{\mu}\dot{X}^{\nu}\,\mathrm{e}^{\mathrm{i}k_3X}(y_3) :: \dot{X}^{\lambda}\dot{X}^{\rho}\,\mathrm{e}^{\mathrm{i}k_4X}(y_4) :> \\ &= \int \mathrm{d}\mu(y_i)\prod_{i<j}|y_i-y_j|^{2\alpha' k_i k_j} \\ &\Big\{(F^{\mu}F^{\nu})_{y_3}(F^{\lambda}F^{\rho})_{y_4} + (-2\alpha')\Big[F^{\mu}_{y_3}F^{\lambda}_{y_4}\frac{\eta^{\nu\rho}}{(y_3-y_4)^2} + F^{\mu}_{y_3}F^{\rho}_{y_4}\frac{\eta^{\nu\lambda}}{(y_3-y_4)^2} \\ &+ F^{\nu}_{y_3}F^{\lambda}_{y_4}\frac{\eta^{\mu\rho}}{(y_3-y_4)^2} + F^{\nu}_{y_3}F^{\rho}_{y_4}\frac{\eta^{\mu\lambda}}{(y_3-y_4)^2}\Big] \\ &+ (-2\alpha')^2\Big[\frac{\eta^{\mu\lambda}}{(y_3-y_4)^2}\frac{\eta^{\nu\rho}}{(y_3-y_4)^2} + \frac{\eta^{\mu\rho}}{(y_3-y_4)^2}\frac{\eta^{\nu\lambda}}{(y_3-y_4)^2}\Big]\Big\}. \end{aligned} \quad (4.1)$$

We shall fix the integration measure by the choice $y_4 = 0$, $y_2 = 1$, $y_1 \to \infty$. The mass-shell conditions are

$$\alpha' k_1^2 = \alpha' k_2^2 = +1, \qquad \alpha' k_3^2 = \alpha' k_4^2 = -1$$

and $k_1 + k_2 + k_3 + k_4 = 0$. Thus

$$\begin{aligned} \int \mathrm{d}\mu(y_i)\prod_{i<j}|y_i-y_j|^{2\alpha' k_i k_j} &= \int_{-\infty}^{+\infty}\mathrm{d}y_3\, y_1^2 \prod_{y_4=0,\,y_2=1,\,y_1\to\infty}^{i<j}|y_i-y_j|^{2\alpha' k_i k_j} \to \\ &= \int_{-\infty}^{+\infty}\mathrm{d}y_3|y_3|^{2\alpha' k_3k_4}|1-y_3|^{2\alpha' k_2k_3} \end{aligned} \quad (4.2)$$

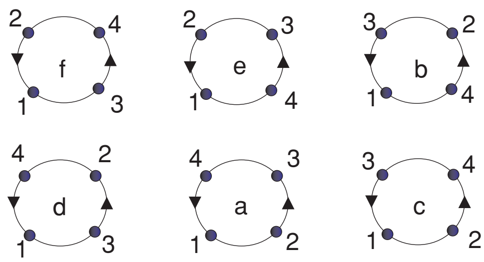

Figure 4. Orderings of four open string vertex operators on the disc. The integral over y_3 in (4.2) splits into three ranges $[-\infty, 0]$, $[0, 1]$, $[1, +\infty]$. For these three ranges the vertex operators are ordered as in (d), (a) and (c), respectively. For the reversed cyclic permutation we shall get (f), (e) and (b). The coordinate y_3 increases in the direction of the arrow.

and

$$
\begin{aligned}
F^{\mu}_{y_3} &= -2\mathrm{i}\alpha' \left(\frac{k_1^{\mu}}{y_3 - y_1} + \frac{k_2^{\mu}}{y_3 - y_2} + \frac{k_4^{\mu}}{y_3 - y_4} \right) \rightarrow -2\mathrm{i}\alpha' \left(\frac{k_2^{\mu}}{y_3 - 1} + \frac{k_4^{\mu}}{y_3} \right), \\
F^{\mu}_{y_4} &= -2\mathrm{i}\alpha' \left(\frac{k_1^{\mu}}{y_4 - y_1} + \frac{k_2^{\mu}}{y_4 - y_2} + \frac{k_3^{\mu}}{y_4 - y_3} \right) \rightarrow -2\mathrm{i}\alpha' \left(-k_2^{\mu} - \frac{k_3^{\mu}}{y_3} \right).
\end{aligned}
\tag{4.3}
$$

Therefore we shall get

$$
\begin{aligned}
F^{\mu\nu,\lambda\rho}(k_1, k_2, k_3, k_4) &= \int_{-\infty}^{+\infty} \mathrm{d}y_3 |y_3|^{2\alpha' k_3 k_4} |1 - y_3|^{2\alpha' k_2 k_3} \\
&\left\{ (2\alpha')^4 \left(\frac{k_2^{\mu}}{y_3 - 1} + \frac{k_4^{\mu}}{y_3} \right) \left(\frac{k_2^{\nu}}{y_3 - 1} + \frac{k_4^{\nu}}{y_3} \right) \left(k_2^{\lambda} + \frac{k_3^{\lambda}}{y_3} \right) \left(k_2^{\rho} + \frac{k_3^{\rho}}{y_3} \right) \right. \\
&\quad - (2\alpha')^3 \frac{1}{y_3^2} \left[\left(\frac{k_2^{\mu}}{y_3 - 1} + \frac{k_4^{\mu}}{y_3} \right) \left(\left(k_2^{\rho} + \frac{k_3^{\rho}}{y_3} \right) \eta^{\nu\lambda} + \left(k_2^{\lambda} + \frac{k_3^{\lambda}}{y_3} \right) \eta^{\nu\rho} \right) \right. \\
&\quad \left. + \left(\frac{k_2^{\nu}}{y_3 - 1} + \frac{k_4^{\nu}}{y_3} \right) \left(\left(k_2^{\lambda} + \frac{k_3^{\lambda}}{y_3} \right) \eta^{\mu\rho} + \left(k_2^{\rho} + \frac{k_3^{\rho}}{y_3} \right) \eta^{\mu\lambda} \right) \right] \\
&\quad \left. + (2\alpha')^2 \frac{1}{y_3^4} [\eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda}] \right\}.
\end{aligned}
\tag{4.4}
$$

The integration over y_3 can be divided into three regions: $[-\infty, 0]$, $[0, 1]$, $[1, +\infty]$. These pieces can be depicted by three diagrams (d) (a) and (c) in figure 4. For the reversed cyclic permutation, we shall get diagrams (f), (e) and (b) in figure 4. Introducing Mandelshtam variables,

$$
s = -(k_1 + k_2)^2, \qquad t = -(k_2 + k_3)^2, \qquad u = -(k_2 + k_4)^2,
$$

we can represent the contribution of the (s, t) diagrams (a) and (e) in the form

$$
\begin{aligned}
&-(2\alpha')^3 (\mathrm{tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4}) + \mathrm{tr}(\lambda^{a_4}\lambda^{a_3}\lambda^{a_2}\lambda^{a_1})) \\
&[B(-\alpha' s, -\alpha' t + 1) K^{\mu\rho\nu\lambda}(k_4, k_2) + B(-\alpha' s - 1, -\alpha' t + 1) K^{\mu\rho\nu\lambda}(k_4, k_3) \\
&\quad - B(-\alpha' s + 1, -\alpha' t) K^{\mu\rho\nu\lambda}(k_2, k_2) - B(-\alpha' s, -\alpha' t) K^{\mu\rho\nu\lambda}(k_2, k_3)],
\end{aligned}
\tag{4.5}
$$

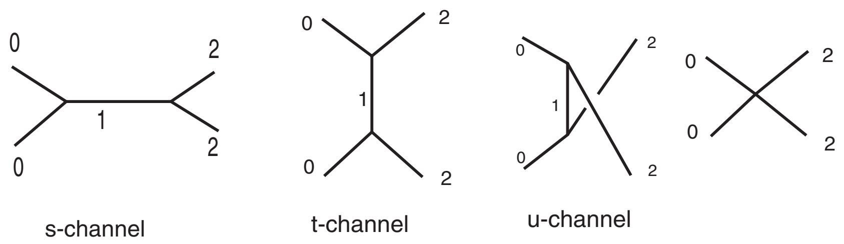

Figure 5. Feynman diagrams contributing to the low-energy limit of the open string scattering amplitude. The poles correspond to the exchange by a massless vector boson in s, t and u channels. The scalars are depicted as 0, vectors are as 1 and tensors are as 2. The quantities of main interest here are the dimensionless vertices $\mathcal{V}_{1-2-2}$ between a vector and two tensors in the s-channel and $\mathcal{V}_{0-1-2}$ between a scalar, vector and a tensor in t, u channels. The last graph represents the contact vertex $\mathcal{V}_{0-0-2-2}$.

the contribution of the (s, u) diagrams (f) and (c) in the form

$$-(2\alpha')^3(\mathrm{tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3}) + \mathrm{tr}(\lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2}))$$
$$[-B(-\alpha's, -\alpha'u)K^{\mu\rho\nu\lambda}(k_4, k_2) + B(-\alpha's - 1, -\alpha'u + 1)K^{\mu\rho\nu\lambda}(k_4, k_3)$$
$$- B(-\alpha's + 1, -\alpha'u)K^{\mu\rho\nu\lambda}(k_2, k_2) + B(-\alpha's, -\alpha'u + 1)K^{\mu\rho\nu\lambda}(k_2, k_3)], \tag{4.6}$$

and the contribution of the (u, t) diagrams (b) and (d) in the form

$$-(2\alpha')^3(\mathrm{tr}(\lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\lambda^{a_4}) + \mathrm{tr}(\lambda^{a_1}\lambda^{a_4}\lambda^{a_2}\lambda^{a_3}))$$
$$[B(-\alpha'u, -\alpha't + 1)K^{\mu\rho\nu\lambda}(k_4, k_2) + B(-\alpha'u + 1, -\alpha't + 1)K^{\mu\rho\nu\lambda}(k_4, k_3)$$
$$+ B(-\alpha'u, -\alpha't)K^{\mu\rho\nu\lambda}(k_2, k_2) + B(-\alpha'u + 1, -\alpha't)K^{\mu\rho\nu\lambda}(k_2, k_3)], \tag{4.7}$$

where

$$K^{\mu\rho\nu\lambda}(k, p) = k^\mu(p^\rho\eta^{\nu\lambda} + p^\lambda\eta^{\nu\rho}) + k^\nu(p^\lambda\eta^{\mu\rho} + p^\rho\eta^{\mu\lambda}). \tag{4.8}$$

Considering the limit $\alpha's, \alpha't, \alpha'u \to 0$ of the Euler functions we shall get for the s , t and u channel contributions:

$$-(2\alpha')^3\left\{+\frac{1}{\alpha's}\,\mathrm{tr}[\lambda^{a_1}, \lambda^{a_2}][\lambda^{a_3}\lambda^{a_4}](K(k_2, k_3) - K(k_4, k_2))^{\mu\rho\nu\lambda}\right.$$
$$- \frac{1}{\alpha't}\,\mathrm{tr}[\lambda^{a_1}, \lambda^{a_4}][\lambda^{a_2}\lambda^{a_3}](K(k_2, k_2) + K(k_2, k_3))^{\mu\rho\nu\lambda}$$
$$\left.- \frac{1}{\alpha'u}\,\mathrm{tr}[\lambda^{a_1}, \lambda^{a_3}][\lambda^{a_2}\lambda^{a_4}](K(k_2, k_2) - K(k_4, k_2))^{\mu\rho\nu\lambda}\right\}, \tag{4.9}$$

which are shown in figure 5.

The last term in the matrix element (4.4) has no momentum dependence and represents the contact term. Evaluating the integration over y_3 in the same way as we did above one can get

$$(2\alpha')^2\{+(\mathrm{tr}\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4} + \mathrm{tr}\lambda^{a_4}\lambda^{a_3}\lambda^{a_2}\lambda^{a_1})B(-\alpha's - 1, -\alpha't + 1)$$
$$+ (\mathrm{tr}\lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3} + \mathrm{tr}\lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2})B(-\alpha's - 1, -\alpha'u + 1)$$
$$+ (\mathrm{tr}\lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\lambda^{a_4}) + \mathrm{tr}\lambda^{a_1}\lambda^{a_4}\lambda^{a_2}\lambda^{a_3})B(-\alpha't + 1, -\alpha'u + 1)\}$$
$$(\eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda}). \tag{4.10}$$

Again, considering the limit $\alpha' s, \alpha' t, \alpha' u \to 0$ of the Euler functions we shall get for the contact term

$$(2\alpha')^2\{\mathrm{tr}[\lambda^{a_1}, \lambda^{a_4}][\lambda^{a_2}, \lambda^{a_3}] + \frac{t}{s}\,\mathrm{tr}[\lambda^{a_1}, \lambda^{a_2}][\lambda^{a_3}, \lambda^{a_4}]\}(\eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda}). \tag{4.11}$$

This is the quartic vertex of $\mathcal{V}_{0-0-2-2}$ and should be compared with the corresponding vertex in the non-Abelian tensor gauge field theory [27–29].

5. Conclusion

Our intention in this paper was to compare the structure of the tree-level scattering amplitudes in non-Abelian tensor gauge field theory and in open string theory with Chan–Paton charges. We limit ourselves to considering only lower-rank tensor fields in both theories. We identify the symmetric T_S and antisymmetric T_A components of the second-rank tensor gauge field $A^a_{\alpha\acute{\alpha}}$ with the string excitations T_S and T_A on the second and third levels. In the process of this identification, we select only those parts of the tree-level scattering amplitudes in the open string theory which are *linear in momentum*.

It should be mentioned that not all three-point string scattering amplitudes have linear in momentum parts. In particular, the scattering amplitude of three tensor bosons does not. It seems that this subclass of tree-level scattering amplitudes may provide information about the structure of the open string theory at the deepest level.

It is also true that tensor gauge field theory bosons are massless while in string theory the tensor bosons are massive. In this respect one could mention that the cubic and quartic vertices of the massless Yang–Mills theory and of the massive theory with spontaneous symmetry breaking are identical. Therefore, if we want to extract a genuine massless 'proto-theory' from the open string theory, which in accordance with Gross is in a broken phase [4], it seems very natural to consider the above subclass of amplitudes and compare them with the amplitudes in non-Abelian tensor gauge field theory.

It is also interesting to mention that the ratio of the masses of the tensor gauge bosons T_S and T_A in non-Abelian tensor gauge theory with spontaneous symmetry breaking is [54]

$$\frac{m_A^2}{m_S^2} = 3, \tag{5.1}$$

while in the open string theory it is (see figure 1)

$$\frac{m_A^2}{m_S^2} = 4. \tag{5.2}$$

In both theories the antisymmetric tensor is more heavy.

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